Formalizing accessibility and duality in a virtual equipment

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← Today's slides

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Φ: a nice class of shapes of colim.

Definition

The *free cocompletion* of $\mathbf{A}$ under filtered colimits $\cdots$ fullsub $\mathbf{Ind}(\mathbf{A}) := \{\text{fil.colim of repr}\} \subseteq \mathbf{Set}^{\mathbf{A}^{\mathrm{op}}}$ (= ind-completion of $\mathbf{A}$.)

Definition

The *free cocompletion* of $\mathbf{A}$ under Φ-colimits $\cdots$ fullsub $\mathbf{Ind}_{\Phi}(\mathbf{A}) := \{\Phi\text{-colim of repr}\} \subseteq \mathbf{Set}^{\mathbf{A}^{\mathrm{op}}}$ (= "Φ-ind-completion" of $\mathbf{A}$.)

Definition

$X \in \mathbf{X}$ is finitely presentable (f.p.) $\overset{\text{def}}{\Leftrightarrow}$ $\mathbf{X}(X, -)$ preserves filtered colimits.

Definition

$X \in \mathbf{X}$ is Φ-atomic $\overset{\text{def}}{\Leftrightarrow}$ $\mathbf{X}(X, -)$ preserves Φ-colimits.

Fact

TFAE for a category $\mathbf{X}$:

1. $\mathbf{X}$ has filtered colimits, and every $X \in \mathbf{X}$ is a filtered colimit of f.p.objects.
2. $\mathbf{X} \simeq \mathbf{Ind}(\mathbf{A})$ $(\exists \mathbf{A})$.

$\Updownarrow$def (if we ignore "size.")

$\mathbf{X}$ is finitely accessible.

Fact

TFAE for a category $\mathbf{X}$:

1. $\mathbf{X}$ has Φ-colimits, and every $X \in \mathbf{X}$ is a Φ-colimit of Φ-atomic obj.
2. $\mathbf{X} \simeq \mathbf{Ind}_{\Phi}(\mathbf{A})$ $(\exists \mathbf{A})$.

$\Updownarrow$def (if we ignore "size.")

$\mathbf{X}$ is Φ-accessible.

Duality

Φ: a nice class of shapes of colim.

Definition (only for today)

A functor $\mathbf{X} \xrightarrow{F} \mathbf{Y}$ is Φ-weighty
$\overset{\text{def}}{\Leftrightarrow}$ (Pointwise) left Kan extensions along F are computed as Φ-colimits.

Theorem (Duality in the Φ-accessible context)

There is a biequivalence of 2-categories:

$$\mathscr{C}au_{\Phi}^{\mathrm{co}} \quad \simeq_{\mathsf{bi}} \quad \mathscr{A}cc_{\Phi}^{\mathrm{op}}$$

The 2-category $\mathscr{C}au_{\Phi}$:

- 0-cell $\cdots$ Cauchy complete small category
- 1-cell $\cdots$ Φ-weighty functor
- 2-cell $\cdots$ natural transformation

The 2-category $\mathscr{A}cc_{\Phi}$:

- 0-cell $\cdots$ Φ-accessible category
- 1-cell $\cdots$ Φ-cocontinuous right adjoint functor
- 2-cell $\cdots$ natural transformation

This is a "Φ-modified" version of *Makkai–Paré duality* (Makkai and Paré 1989).
This duality has recently been generalized to the enriched context (Tendas 2023).

Goal

($\mathscr{V}$-enriched) accessibility

- duality
- ind-completion
- Cauchy completeness

$=$ Accessibility in $\mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$

The *virtual equipment* $\mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$:

- $\mathscr{V}$-enriched categories
- $\mathscr{V}$-functors
- $\mathscr{V}$-profunctors

Formal accessibility in a virtual equipment

$$\mathscr{V}\text{-}\mathbb{P}\mathrm{rof} \xrightarrow{\text{generalize}} \mathbb{E} \text{ (an arbitrary virtual equipment)}$$

This extends the notion of accessibiliy to other category-theoretic contexts:

- bicategory-enriched categories
- fibered (or indexed) categories
- internal categories
- something that is no longer categories

Why virtual equipments?

2-categories are suitable for capturing:

- ✓ ordinary limits and colimits,
- ✓ adjunctions,
- ✓ monads,
- ✓ Kan extensions and lifts.

2-categories are <u>not</u> suitable for capturing interactions of functors and profunctors: $\rightsquigarrow$ *virtual equipments*

- × <u>weighted</u> limits and colimits,
- × presheaves,
- × cocompletions,
- × <u>pointwise</u> Kan extensions,
- × Cauchy completeness,
- × commutation of weights.

Main features

In our formalization,

- We do not use *opposite categories*.
  - $\rightsquigarrow$ categories enriched over a non-symmetric monoidal category or a bicategory
- We do not require either the *smallness* of categories or the *compositions* of arbitrary profunctors.
  - $\rightsquigarrow$ bypassing the *size matters*
- We do not demand "(co)completeness" for the universe.
  - $\rightsquigarrow$ enrichment by a monoidal category that is neither (co)complete nor closed.

The augmented virtual double category $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

- $\mathscr{V}$-categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$;
- $\mathscr{V}$-functors $\begin{tikzcd} \mathbf{A} \arrow[d, "F"'] \\ \mathbf{B} \end{tikzcd}$, . . . and their compositions and identities;
- $\mathscr{V}$-profunctors $\mathbf{A} \xrightarrow{P} \mathbf{B}$, . . . ;
- $\binom{n}{1}$-$\mathscr{V}$-forms $\begin{tikzcd} \mathbf{A}_0 \arrow[r, "P_1"] \arrow[d, "F"'] & \mathbf{A}_1 \arrow[r, "P_2"] & \cdots \arrow[r, "P_n"] \arrow[d, phantom, "\alpha"] & \mathbf{A_n} \arrow[d, "G"] \\ \mathbf{B} \arrow[rrr, "Q"'] & & & \mathbf{C} \end{tikzcd} = \{P_1(A_0, A_1) \otimes \cdots \otimes P_n(A_{n-1}, A_n) \to Q(FA_0, GA_n)\}$,

  $\binom{n}{0}$-$\mathscr{V}$-forms $\begin{tikzcd} \mathbf{A}_0 \arrow[r, "P_1"] \arrow[dr, "F"'] & \cdots \arrow[r, "P_n"] \arrow[d, phantom, "\alpha"] & \mathbf{A_n} \arrow[dl, "G"] \\ & \mathbf{B} & \end{tikzcd} = \{P_1(A_0, A_1) \otimes \cdots \otimes P_n(A_{n-1}, A_n) \to \mathbf{B}(FA_0, GA_n)\}$,
- and their compositions $\begin{tikzcd} \mathbf{A}_0 \arrow[r, dashed, "\vec{P}_1"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\alpha_1"] & \mathbf{A}_1 \arrow[r, dashed, "\vec{P}_2"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{P}_n"] \arrow[dr, phantom, "\alpha_n"] & \mathbf{A}_n \arrow[d, "F_n"] \\ \mathbf{B}_0 \arrow[r, dotted, "Q_1"'] \arrow[d, "G"'] & \mathbf{B}_1 \arrow[r, dotted, "Q_2"'] & \cdots \arrow[r, dotted, "Q_n"'] \arrow[d, phantom, "\beta"] & \mathbf{B}_n \arrow[d, "H"] \\ \mathbf{C} \arrow[rrr, dotted, "R"'] & & & \mathbf{D} \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathbf{A}_0 \arrow[r, "\vec{P}_1"] \arrow[d, "F_0 ⨾ G"'] & \mathbf{A}_1 \arrow[r, "\vec{P}_2"] \arrow[d, phantom, "\vec{\alpha} ⨾ \beta"] & \cdots \arrow[r, "\vec{P}_n"] & \mathbf{A}_n \arrow[d, "F_n ⨾ H"] \\ \mathbf{C} \arrow[rrr, dotted, "R"'] & & & \mathbf{D} \end{tikzcd}$

An augmented virtual double category $\mathbb{X}$ (Koudenburg 2020)

- objects $A, B, C, \ldots$;
- vertical arrows $\begin{array}{c} A \\ {\scriptstyle f}\downarrow \\ B \end{array}$, ... and their compositions and identities;
- horizontal arrows $A \xrightarrow{p} B$, ...;
- cells: $\binom{n}{1}$-cells

$$\begin{tikzcd} A_0 \arrow[r, "p_1", "\shortmid"{marking}] \arrow[d, "f"'] & A_1 \arrow[r, "p_2", "\shortmid"{marking}] & \cdots \arrow[r, "p_n", "\shortmid"{marking}] & A_n \arrow[d, "g"] \\ B \arrow[rrr, "q"', "\shortmid"{marking}] \arrow[rrr, phantom, shift left=4ex, "\alpha"] & & & C \end{tikzcd}, \ldots$$

$\binom{n}{0}$-cells

$$\begin{tikzcd} A_0 \arrow[r, "p_1", "\shortmid"{marking}] \arrow[dr, "f"'] & \cdots \arrow[r, "p_n", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & A_n \arrow[dl, "g"] \\ & B & \end{tikzcd}, \ldots$$

- and their compositions:

$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{p}_1", "\shortmid"{marking}] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A_1 \arrow[r, dashed, "\vec{p}_2", "\shortmid"{marking}] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, dashed, "\vec{p}_n", "\shortmid"{marking}] \arrow[d, phantom, "\alpha_n"] & A_n \arrow[d, "f_n"] \\ B_0 \arrow[r, dotted, "q_1", "\shortmid"{marking}] \arrow[d, "g"'] & B_1 \arrow[r, dotted, "q_2", "\shortmid"{marking}] \arrow[d, phantom, "\beta"] & \cdots \arrow[r, dotted, "q_n", "\shortmid"{marking}] & B_n \arrow[d, "h"] \\ C \arrow[rrr, dotted, "r"', "\shortmid"{marking}] & & & D \end{tikzcd} \rightsquigarrow \begin{tikzcd} A_0 \arrow[r, dashed, "\vec{p}_1", "\shortmid"{marking}] \arrow[d, "f_0 \fatsemi g"'] & A_1 \arrow[r, dashed, "\vec{p}_2", "\shortmid"{marking}] \arrow[d, phantom, "\vec{\alpha} \fatsemi \beta"] & \cdots \arrow[r, dashed, "\vec{p}_n", "\shortmid"{marking}] & A_n \arrow[d, "f_n \fatsemi h"] \\ C \arrow[rrr, dotted, "r"', "\shortmid"{marking}] & & & D \end{tikzcd}$$

(and identity cells.)

Virtual equipments

Definition (Cruttwell and Shulman 2010; Koudenburg 2020)

A virtual equipment $=$ an augmented virtual double category s.t.
$$\begin{tikzcd} A \arrow[r, "\exists p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\exists\text{cart}"] & B \arrow[d, "g"] \\ X \arrow[r, dotted, "u"', "\shortmid"{marking}] & Y \end{tikzcd}$$

Example

virtual equipment	object	vert.arrow	hor.arrow
$\mathscr{V}$-$\mathbb{P}\mathrm{rof}$ ($\mathscr{V}$: a monoidal cat)	$\mathscr{V}$-enriched cat	$\mathscr{V}$-functor	$\mathscr{V}$-profunctor
$\mathscr{W}$-$\mathbb{P}\mathrm{rof}$ ($\mathscr{W}$: a bicategory)	$\mathscr{W}$-enriched cat	$\mathscr{W}$-functor	$\mathscr{W}$-profunctor
$\mathbb{P}\mathrm{rof}(\mathbf{C})$ ($\mathbf{C}$: cat with p.b.)	$\mathbf{C}$-internal cat	$\mathbf{C}$-internal functor	$\mathbf{C}$-internal profunctor

and so on.

From now on, we fix a virtual equipment $\mathbb{E}$. (e.g. $\mathbb{E} := \mathscr{V}$-$\mathbb{P}\mathrm{rof}$)

Zoo of cells

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

$$\begin{tikzcd} A_0 \arrow[r, dashed, "\vec{u}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\text{comp}"] & A_n \arrow[d, equal] \\ A_0 \arrow[r, "|" marking] & A_n \end{tikzcd} \quad : \text{composing} \qquad \left(\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\text{comp}"] & \\ A \arrow[rr, equal, "|" marking, "\mathrm{Id}_A"'] & & A \end{tikzcd} \right)$$

$\rightsquigarrow$

- composition of profunctors,
- colimits in $\mathscr{V}$.

$$\begin{tikzcd} C \arrow[r, "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\text{lift}"] & B \arrow[r, dashed, "\vec{u}", "|" marking] & A \arrow[d, "f"] \\ C \arrow[rr, dotted, "v"', "|" marking] & & X \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "|" marking] \arrow[d, "f"'] \arrow[drr, phantom, "\text{ext}"] & B \arrow[r, "|" marking] & C \arrow[d, equal] \\ X \arrow[rr, dotted, "v"', "|" marking] & & C \end{tikzcd}$$

$: \text{lifting} \qquad\qquad : \text{extending}$

$\rightsquigarrow$

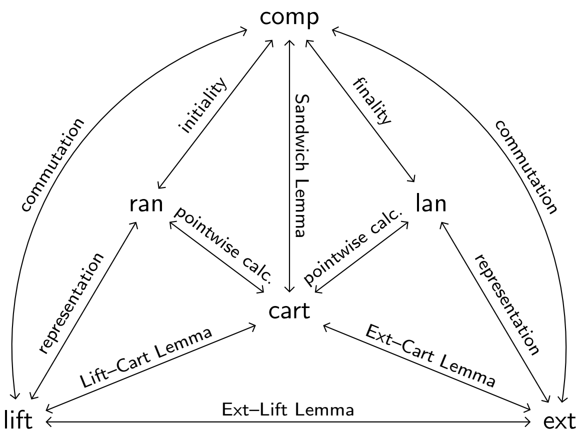
- lift/extension of profunctors,
- limits in $\mathscr{V}$.

$$\begin{tikzcd} B \arrow[r, "u", "|" marking] \arrow[dr] & A \arrow[d, "f"] \arrow[dl, phantom, "\text{ran}"] \\ & X \end{tikzcd} \quad : \text{right Kan ext.} \qquad \begin{tikzcd} A \arrow[r, "u", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\text{lan}"] & B \arrow[dl] \\ X & \end{tikzcd} \quad : \text{left Kan ext.}$$

$\rightsquigarrow$

- weighted limits/colimits,
- pointwise Kan extensions.

Techniques in a virtual equipment



Formal category theory in a virtual equipment
$=$ A *puzzle* to be solved using lemmas and relationships like above.

(Co)completeness and (co)continuity

A left weight $\cdots$ A horizontal arrow $A \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B$ satisfying a "nice" property.
We regard A as a "shape of diagrams," and φ as "weights parametrized by B."

Definition

Φ: a class of left weights

1. X is Φ-cocomplete $\overset{\text{def}}{\Leftrightarrow}$
$$\begin{tikzcd} A \arrow[r, "\varphi", "+"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{lan}" near start] & B \arrow[dl, "\exists"] \\ X & \end{tikzcd}$$
$(\forall \varphi \in \Phi,\ \forall f)$

2. $\begin{matrix} X \\ g\downarrow \\ Y \end{matrix}$ is Φ-cocontinuous $\overset{\text{def}}{\Leftrightarrow}$
$$\begin{tikzcd} A \arrow[r, "\varphi", "+"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\mathsf{lan}" near start] & B \arrow[dl] \\ X \arrow[d, "g"'] & \\ Y & \end{tikzcd}$$
is a lan-cell $(\forall \varphi \in \Phi,\ \forall f)$.

A class Φ of left weights plays a role as a class of "shapes" of colimits.

In $\mathscr{V}$-$\mathbb{P}$rof,

$$\begin{array}{l} \mathbf{A} \\ {\scriptstyle \text{yoneda}}\downarrow \\ \mathbf{Ind}_\Phi(\mathbf{A}) = \{\varphi \colon \mathbf{A}^{\mathrm{op}} \to \mathscr{V} \quad \text{in } \Phi\} \end{array} \quad : \text{the free cocompletion of } \mathbf{A} \text{ under } \Phi.$$

Then,
$$\begin{array}{c} \mathbf{B} \\ {\scriptstyle F}\downarrow \\ \mathbf{Ind}_\Phi(\mathbf{A}) \end{array} \quad \Bigg\| \quad \mathbf{A}^{\mathrm{op}} \otimes \mathbf{B} \xrightarrow{F} \mathscr{V} \text{ s.t. } F(-, \forall b) \in \Phi \quad \Bigg\| \quad \mathbf{A} \overset{F}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathbf{B} \quad \text{in } \Phi$$

Definition

Φ: a nice class of left weights in $\mathbb{E}$.

$\begin{array}{c} A \\ {\scriptstyle k}\downarrow \\ X \end{array}$ is a Φ-ind-morphism $\overset{\text{def}}{\Leftrightarrow}$ k yields adj equiv: $\mathbf{Hom}_{\mathbb{E}}\left(\begin{smallmatrix} B \\ X \end{smallmatrix}\right) \underset{\mathrm{Lan}_{-}k}{\overset{X(k,-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathbf{Hom}_\Phi(A, B)$. $\quad (\forall B \in \mathbb{E})$

Remark

Φ-ind-morphisms are a Φ-modified version of *Yoneda morphisms* in the sense of (Koudenburg 2022).

Notation

$A \longrightarrow \Phi^\nabla A$: a Φ-ind-morphism. (up to vertical equivalences)

The “functor” $A \mapsto \Phi^\nabla A$

Question

- Does the assignment $A \mapsto \Phi^\nabla A$ yield a “functor”?
- What are the domain and codomain of Φ^∇?
- Does Φ^∇ have a universal property?

$\Phi^\triangledown$ behaves like a left adjoint

Observation 1

$$\frac{A \xrightarrow{\varphi} UB \quad \text{in } \Phi}{\Phi^\triangledown A \xleftarrow{\hat{\varphi}} B} \text{ (by def. of } \Phi\text{-ind-mor.)}$$

Observation 2

$$\begin{array}{c} A \\ f\downarrow \\ UB \end{array} \Bigg\| \begin{array}{c} \Phi^\triangledown A \\ \downarrow \hat{f}\text{: } \Phi\text{-cocts} \\ B \end{array} \quad (\Phi^\triangledown \text{ is a "}\Phi\text{-cocompletion."})$$

$\rightsquigarrow \quad \Phi^\triangledown \dashv U$? $\qquad (U : B \mapsto B)$

Definition

Φ: a nice class of left weights in $\mathbb{E}$.

The pseudo-double category $\mathbb{E}_\Phi$:

- object $\cdots$ the same as $\mathbb{E}$
- vert.arrow $\cdots$ the same as $\mathbb{E}$
- hor.arrow $\cdots$ hor.arrow in Φ
- cell $\cdots$ the same as $\mathbb{E}$

The (strict) dbl cat $\mathbb{Q}\Phi^\triangledown$ (*quintet-like const.*):

- object $\cdots$ Φ-cocomplete object in $\mathbb{E}$
- vert.arrow $\cdots$ Φ-cocts vert.arrow in $\mathbb{E}$
- hor.arrow $X \nrightarrow Y$ $\cdots$ vert.arrow $X \leftarrow Y$
- cell $\begin{array}{ccc} X & \overset{u}{\nrightarrow} & Y \\ f\downarrow & \alpha & \downarrow g \\ Z & \underset{v}{\nrightarrow} & W \end{array}$ $\cdots$ $\begin{array}{ccccc} & & Y & & \\ & \overset{u}{\swarrow} & & \overset{g}{\searrow} & \\ X & & \alpha & & W \\ & \underset{f}{\searrow} & & \underset{v}{\swarrow} & \\ & & Z & & \end{array}$ in $\mathbb{E}$

Relative companied biadjoints and duality

$$\begin{tikzcd} \mathbb{E}_\Phi{}' \arrow[r, phantom, "\subseteq"] \arrow[dr, "\Phi^\triangledown"'] & \mathbb{E}_\Phi \arrow[d, phantom, "\Searrow\eta"] \\ & \mathbb{Q}\Phi^\triangledown \arrow[u, "U"'] \end{tikzcd}$$

$\Phi^\triangledown$ behaves like a (relative) left adjoint of U w.r.t. the vertical and horizontal directions.

$\rightsquigarrow$ forms a relative companied biadjunction. (a new concept)

Theorem (Duality)

There is an "equivalence" of double categories:

$$\begin{array}{ccccc} & \mathbb{C}\mathrm{au}\Phi^\triangledown & \simeq & \mathbb{A}\mathrm{cc}\Phi^\triangledown & \\ & \cap\!| & & \cap\!| & \\ (& \mathbb{E}_\Phi{}' & \xrightarrow{\Phi^\triangledown} & \mathbb{Q}\Phi^\triangledown &) \end{array}$$

The (strict) double category $\mathbb{A}\mathrm{cc}\Phi^\triangledown$:

- obj $\cdots$ objects in the image of $\Phi^\triangledown$ (Φ-accessible obj)
- vert.arr $\cdots$ Φ-cocts right adjoint vert.arrow
- hor.arr $\cdots$ Φ-cocts vert.arrow (in the opposite direction)

The (pseudo) double category $\mathbb{C}\mathrm{au}\Phi^\triangledown$:

- obj $\cdots$ A "Cauchy cpl" obj $A \in \mathbb{E}$ s.t. $\exists\ \Phi^\triangledown A$
- vert.arr $\cdots$ A vert.arrow f in $\mathbb{E}$ s.t. $f_* \in \Phi$
- hor.arr $\cdots$ that in Φ

Ongoing works

- Developing a formal theory of “locally presentable objects”
- Exploring virtual equipments $\mathbb{E}$ that provide interesting duality.
- Comparing with related work: formal accessibility in a 2-category with a “KZ context” (Di Liberti and Loregian 2023)

Thank you!

Today's slides

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Commutation of limits and colimits

Commutation in $\mathbf{Set}$

Φ: a class of "shapes" of colim, Ψ: a class of "shapes" of lim.

In $\mathbf{Set}$,

Φ-colimits	= colim commuting with	Ψ-limits
filtered colimits		finite limits
κ-filtered colimits		κ-limits
sifted colimits		finite products
connected colimits		terminal
coproducts of filtered colimits		finite connected limits
absolute colimits		small limits
small colimits		"nothing"

$\Psi_{/\!/}$: the class of "shapes" of colim commuting with Ψ-lim in $\mathbf{Set}$.

- finitely accessible = $\Psi_{/\!/}$-accessible (Ψ: finite limits)
- κ-accessible = $\Psi_{/\!/}$-accessible (Ψ: κ-limits)
- generalized variety = $\Psi_{/\!/}$-accessible (Ψ: finite products) (Adámek and Rosický 2001)

Theorem

If Ψ satisfies a "nice" condition and $\mathbf{A}$: Ψ-cocomplete, then

$\mathbf{A} \xrightarrow{F} \mathbf{B}$ is $\Psi_{/\!/}$-weighty $\Leftrightarrow$ $\mathbf{A} \xrightarrow{F} \mathbf{B}$ is Ψ-cocontinuous

Definition (only for today)

$\mathbf{X}$ is locally Ψ-presentable $\overset{\text{def}}{\Leftrightarrow}$ it is a $\Psi_{/\!/}$-ind-completion of Cauchy cpl $\wedge$ Ψ-cocpl small cat.

Theorem (Duality for the locally Ψ-presentable context)

If Ψ satisfies a "nice" condition,

$$\begin{array}{ccc} \mathscr{C}oth_{\Psi}^{\mathrm{co}} & \simeq_{\text{bi}} & \mathscr{L}p_{\Psi}^{\mathrm{op}} \\ \cap & & \cap \\ (\ \mathscr{C}au_{\Psi_{/\!/}}^{\mathrm{co}} & \simeq_{\text{bi}} & \mathscr{A}cc_{\Psi_{/\!/}}^{\mathrm{op}} \ ) \end{array}$$

The 2-category $\mathscr{C}oth_{\Psi}$:

- 0-cell ⋯ Cauchy cpl $\wedge$ Ψ-cocpl small cat
- 1-cell ⋯ Ψ-cocontinuous functor
- 2-cell ⋯ natural transformation

The 2-category $\mathscr{L}p_{\Psi}$:

- 0-cell ⋯ locally Ψ-presentable category
- 1-cell ⋯ $\Psi_{/\!/}$-cocts right adjoint functor
- 2-cell ⋯ natural transformation

This subsumes *Gabriel–Ulmer duality* (Ψ=fin.lim), *Adamek–Lawvere–Rosický duality* (Ψ=fin.products).

Compositions

Definition

1.
$$
\begin{tikzcd}
A'_0 \arrow[r, "\vec{u}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, "\vec{u}_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, "\vec{u}_n"] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\
A_0 \arrow[r, dotted, "v_1"'] & A_1 \arrow[r, dotted, "v_2"'] & \cdots \arrow[r, dotted, "v_n"'] & A_n
\end{tikzcd}
\quad \text{is opcartesian}
$$

$$
\overset{\text{def}}{\Leftrightarrow}
\begin{tikzcd}
A'_0 \arrow[r, "\vec{u}_1"] \arrow[d, "f_0"'] & A'_1 \arrow[r, "\vec{u}_2"] & \cdots \arrow[r, "\vec{u}_n"] & A'_n \arrow[d, "f_n"] \\
A_0 \arrow[d, "\forall g"'] & \forall\beta & & A_n \arrow[d, "\forall h"] \\
\forall X \arrow[rrr, dotted, "\forall w"'] & & & \forall Y
\end{tikzcd}
=
\begin{tikzcd}
A'_0 \arrow[r, "\vec{u}_1"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\alpha_1"] & A'_1 \arrow[r, "\vec{u}_2"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\alpha_2"] & \cdots \arrow[r, "\vec{u}_n"] \arrow[dr, phantom, "\alpha_n"] & A'_n \arrow[d, "f_n"] \\
A_0 \arrow[r, dotted, "v_1"'] \arrow[d, "g"'] & A_1 \arrow[r, dotted, "v_2"'] & \exists!\bar{\beta} \quad \cdots \arrow[r, dotted, "v_n"'] & A_n \arrow[d, "h"] \\
X \arrow[rrr, dotted, "w"'] & & & Y
\end{tikzcd}
$$

2.
$$
\begin{tikzcd}
A_0 \arrow[r, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & A_n \arrow[d, equal] \\
A_0 \arrow[r, "v"'] & A_n
\end{tikzcd}
\quad \text{is composing} \quad \overset{\text{def}}{\Leftrightarrow} \quad
\begin{tikzcd}
X \arrow[r, "\vec{p}"] \arrow[d, equal] \arrow[dr, phantom, "="] & A_0 \arrow[r, "\vec{u}"] \arrow[d, equal] \arrow[dr, phantom, "\alpha"] & A_n \arrow[r, "\vec{q}"] \arrow[d, equal] \arrow[dr, phantom, "="] & Y \arrow[d, equal] \\
X \arrow[r, "\vec{p}"'] & A_0 \arrow[r, "v"'] & A_n \arrow[r, "\vec{q}"'] & Y
\end{tikzcd}
\quad \text{is opcartesian.} \ (\forall X, Y, \vec{p}, \vec{q})
$$

Compositions

Example in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

1. $$\begin{tikzcd} \mathbf{A} \arrow[r, "P", "|" marking] \arrow[d, equal] \arrow[drr, phantom, "\text{comp}"] & \mathbf{B} \arrow[r, "Q", "|" marking] & \mathbf{C} \arrow[d, equal] \\ \mathbf{A} \arrow[rr, "P \odot Q"', "|" marking] & & \mathbf{C} \end{tikzcd}$$ $$(P \odot Q)(a, c) := \int^{b \in \mathbf{B}} P(a, b) \otimes Q(b, c) \quad \text{in } \mathscr{V}$$

(Suppose that the above coend is preserved by $X \otimes -, - \otimes Y$.)

2. $$\begin{tikzcd} \mathbf{A} \arrow[rr, equal, "\mathsf{Id}_A", "|" marking] \arrow[dr, equal] \arrow[rr, phantom, shift right=4ex, "\text{cart}"] & & \mathbf{A} \arrow[dl, equal] \\ & \mathbf{A} & \end{tikzcd}$$ $$\mathsf{Id}_A(a, a') := \mathbf{A}(a, a') \quad \text{in } V$$

By universality, $$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] & \\ A \arrow[dr, equal] \arrow[rr, phantom, "="] & & A \arrow[dl, equal] \\ & A & \end{tikzcd} = \begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\exists! \alpha"] & \\ A \arrow[rr, equal, "|" marking] \arrow[dr, equal] \arrow[rr, phantom, shift right=4ex, "\text{cart}"] & {} & A \arrow[dl, equal] \\ & A & \end{tikzcd}$$ $\rightsquigarrow$ α becomes composing.

Ext/Lift

Definition

1. $$\begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "|" marking] \arrow[d, "f"'] & B \arrow[r, "p", "|" marking] \arrow[d, phantom, "\alpha"] & C \arrow[d, equal] \\ X \arrow[rr, dotted, "v"', "|" marking] & & C \end{tikzcd}$$ is extending (We say that p is an *extension* of $(\vec{u}, f, v)$.)

$$\overset{\text{def}}{\Leftrightarrow} \quad \begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "|" marking] \arrow[d, "f"'] & B \arrow[r, dashed, "\vec{q}", "|" marking] \arrow[d, phantom, "\beta"] & Y \arrow[d, "g"] \\ X \arrow[rr, dotted, "v"', "|" marking] & & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, dashed, "\vec{u}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\|"] & B \arrow[r, dashed, "\vec{q}", "|" marking] \arrow[d, equal] \arrow[dr, phantom, "\exists!\bar{\beta}"] & Y \arrow[d, "g"] \\ A \arrow[r, dashed, "\vec{u}", "|" marking] \arrow[d, "f"'] & B \arrow[r, "p", "|" marking] \arrow[d, phantom, "\alpha"] & C \arrow[d, equal] \\ X \arrow[rr, dotted, "v"', "|" marking] & & C \end{tikzcd}$$

2. $$\begin{tikzcd} C \arrow[r, "p", "|" marking] \arrow[d, equal] & B \arrow[r, dashed, "\vec{u}", "|" marking] \arrow[d, phantom, "\alpha"] & A \arrow[d, "f"] \\ C \arrow[rr, dotted, "v"', "|" marking] & & X \end{tikzcd}$$ is lifting (the dual notion of extension)

Ext/Lift

Example in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

1.
$$\begin{tikzcd} \mathbf{A} \arrow[r, "P", "|" marking] \arrow[d, "F"'] & \mathbf{B} \arrow[r, "P\triangleright^F Q", "|" marking] \arrow[d, phantom, "\text{ext}"] & \mathbf{C} \arrow[d, equal] \\ \mathbf{X} \arrow[rr, "Q"', "|" marking] & & \mathbf{C} \end{tikzcd}$$

$$(P \triangleright^F Q)(b,c) := \int_{a\in\mathbf{A}} P(a,b) \triangleright Q(Fa,c) \quad \text{in } \mathscr{V}$$

2.
$$\begin{tikzcd} \mathbf{C} \arrow[r, "Q^F\blacktriangleleft P", "|" marking] \arrow[d, equal] & \mathbf{B} \arrow[r, "P", "|" marking] \arrow[d, phantom, "\text{lift}"] & \mathbf{A} \arrow[d, "F"] \\ \mathbf{C} \arrow[rr, "Q"', "|" marking] & & \mathbf{X} \end{tikzcd}$$

$$(Q^F \blacktriangleleft P)(c,b) := \int_{a\in\mathbf{A}} Q(c,Fa) \blacktriangleleft P(b,a) \quad \text{in } \mathscr{V}$$

Here, $\begin{tikzcd} \mathscr{V} \arrow[r, shift left, "X\otimes -"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[l, shift left, "X\triangleright -"] \end{tikzcd}$, $\quad \begin{tikzcd} \mathscr{V} \arrow[r, shift left, "-\otimes X"] \arrow[r, phantom, "\perp"] & \mathscr{V} \arrow[l, shift left, "-\blacktriangleleft X"] \end{tikzcd}$.

Lan/Ran

Definition (Koudenburg 2022)

1. $$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[dl, "l"] \\ X & \end{tikzcd}$$ is a lan-cell $\overset{\text{def}}{\Leftrightarrow}$ $$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\beta"] & B \arrow[r, dashed, "\vec{v}", "\shortmid"{marking}] & Y \arrow[dll, "g"] \\ X & & \end{tikzcd} = \begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, dashed, "\vec{v}", "\shortmid"{marking}] \arrow[dl, "l"] \arrow[dr, phantom, "\exists!\bar{\beta}"] & Y \arrow[dll, "g"] \\ X & & \end{tikzcd}$$
   (We say that α exhibits l as a *left Kan extension* of f along u.)
2. $$\begin{tikzcd} B \arrow[r, "u", "\shortmid"{marking}] \arrow[dr, "r"'] \arrow[r, phantom, shift right=4ex, "\alpha"] & A \arrow[d, "f"] \\ & X \end{tikzcd}$$ is a ran-cell (the dual notion of lan-cells)

Lemma

$$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[dl, "l"] \\ X & \end{tikzcd}$$ is a lan-cell $\Leftrightarrow$ $$\begin{tikzcd} A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "l_*", "\shortmid"{marking}] \arrow[dl, "l"] \arrow[dr, phantom, "\text{cart}"] & X \arrow[dll, equal] \\ X & & \end{tikzcd}$$ is extending

Lan/Ran

Example in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

1. $$\begin{tikzcd} \mathbf{A} \arrow[r, "W", "|" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\text{lan}"] & \mathbf{1} \arrow[dl, "L"] \\ \mathbf{X} & \end{tikzcd} \quad \Leftrightarrow \quad L* \cong \operatorname*{Colim}_{a \in \mathbf{A}}^{Wa} Fa. \qquad (W\text{-weighted colimit of } F)$$

2. $$\begin{tikzcd} \mathbf{A} \arrow[r, "W", "|" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\text{lan}"] & \mathbf{B} \arrow[dl, "L"] \\ \mathbf{X} & \end{tikzcd} \quad \Leftrightarrow \quad \forall b \in \mathbf{B},\ \ Lb \cong \operatorname*{Colim}_{a \in \mathbf{A}}^{W(a,b)} Fa. \qquad (W(-,b)\text{-weighted colimit of } F)$$

3. $$\begin{tikzcd} \mathbf{A} \arrow[r, "G_*", "|" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\text{lan}"] & \mathbf{B} \arrow[dl, "L"] \\ \mathbf{X} & \end{tikzcd} \quad \Leftrightarrow \quad L \text{ is a } \underline{\text{pointwise}} \text{ left Kan extension of } F \text{ along } G.$$

Lan(ran)-cells subsume <u>pointwise</u> Kan extensions and weighted (co)limits.

Doctrine

Definition

A class Φ of left weights is a left doctrine (or, Φ is *saturated*) $\overset{\text{def}}{\Leftrightarrow}$

- $\mathrm{Id}_A \in \Phi\ (\forall A)$;
- $\varphi, \varphi' \in \Phi \implies \varphi \odot \varphi' \in \Phi$;
- $f^* \in \Phi\ (\forall f)$.

Φ^*: the smallest left doctrine containing Φ

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

- $\mathbf{A} \overset{\psi}{\nrightarrow} \mathbf{B} \in \Phi^* \quad \Leftrightarrow \quad \psi(-, \forall b)$ lies in the itelated closure of $\{\mathrm{rep}\} \subset [\mathbf{A}^{\mathrm{op}}, \mathscr{V}]$ under Φ-colimits.
- Thus, Φ^* is the "*saturation*" of Φ.

Remark

In an arbitrary virtual equipment,

- Φ-cocomplete $\Leftrightarrow$ Φ^*-cocomplete
- Φ-cocontinuous $\Leftrightarrow$ Φ^*-cocontinuous

Commutation

$A_1 \overset{\varphi_1}{\nrightarrow} B_1$ preserves an extension
$$\begin{tikzcd} A_0 \arrow[r, "\varphi_0", "\shortmid"{marking}] \arrow[d, "f"'] & B_0 \arrow[r, "p", "\shortmid"{marking}] \arrow[d, phantom, "\text{ext}"] & A_1 \arrow[d, equal] \\ X \arrow[rr, dotted, "u"', "\shortmid"{marking}] & & A_1 \end{tikzcd}$$

$\overset{\text{def}}{\Leftrightarrow}$ $\exists \alpha, \beta, \gamma$ s.t.

$$\begin{tikzcd} A_0 \arrow[r, "\varphi_0", "\shortmid"{marking}] \arrow[d, "f"'] & B_0 \arrow[r, "p", "\shortmid"{marking}] \arrow[d, phantom, "\text{ext}"] & A_1 \arrow[r, "\varphi_1", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & B_1 \arrow[d, equal] \\ X \arrow[rr, dotted, "u"', "\shortmid"{marking}] \arrow[d, equal] & & A_1 \arrow[r, "\varphi_1", "\shortmid"{marking}] & B_1 \arrow[d, equal] \\ X \arrow[rrr, "\shortmid"{marking}] \arrow[rrr, phantom, shift left=4ex, "\alpha\colon \text{comp}"] & & & B_1 \end{tikzcd} = \begin{tikzcd} A_0 \arrow[r, "\varphi_0", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\shortparallel"] & B_0 \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] & A_1 \arrow[r, "\varphi_1", "\shortmid"{marking}] \arrow[d, phantom, "\beta\colon \text{comp}"] & B_1 \arrow[d, equal] \\ A_0 \arrow[r, "\varphi_0", "\shortmid"{marking}] \arrow[d, "f"'] & B_0 \arrow[rr, "\shortmid"{marking}] \arrow[d, phantom, "\gamma\colon \text{ext}"] & & B_1 \arrow[d, equal] \\ X \arrow[rrr, "\shortmid"{marking}] & & & B_1 \end{tikzcd}$$

Definition

- A pair (φ_0, φ_1) of left weights commutes $(\varphi_0 \mathbin{/\!/} \varphi_1)$ $\overset{\text{def}}{\Leftrightarrow}$ $A_1 \overset{\varphi_1}{\nrightarrow} B_1$ *preserves*
$$\begin{tikzcd} A_0 \arrow[r, "\varphi_0", "\shortmid"{marking}] \arrow[d, "\forall f"'] & B_0 \arrow[r, "\shortmid"{marking}] \arrow[d, phantom, "\text{ext}"] & A_1 \arrow[d, equal] \\ X \arrow[rr, dotted, "\forall u"', "\shortmid"{marking}] & & A_1 \end{tikzcd}$$

- A pair (φ_0, φ_1) of l.w. weakly commutes (φ_0 / φ_1) $\overset{\text{def}}{\Leftrightarrow}$ $A_1 \overset{\varphi_1}{\nrightarrow} B_1$ *preserves*
$$\begin{tikzcd} A_0 \arrow[r, "\varphi_0", "\shortmid"{marking}] \arrow[dr, "\forall f"'] & B_0 \arrow[r, "\shortmid"{marking}] \arrow[d, phantom, "\text{ext}"] & A_1 \arrow[dl, equal] \\ & A_1 & \end{tikzcd}$$

Commutation

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

- $(\mathbf{A} \overset{\varphi}{\nrightarrow} \mathbf{B}) \mathbin{/\!\!/} (\mathbf{C} \overset{\psi}{\nrightarrow} \mathbf{D}) \iff$ φ-limits and ψ-colimits commute in $\mathscr{V}$.
  $\iff [\mathbf{C}, \mathscr{V}] \xrightarrow{\mathrm{Colim}^{\psi(-,d)}} \mathscr{V}$ preserves φ-limits.
- $(\mathbf{A} \overset{\varphi}{\nrightarrow} \mathbf{B}) \mathbin{/} (\mathbf{C} \overset{\psi}{\nrightarrow} \mathbf{D}) \iff [\mathbf{C}, \mathscr{V}] \xrightarrow{\mathrm{Colim}^{\psi(-,d)}} \mathscr{V}$ preserves φ-limits of representables.

Notation

Φ: a class of left weights. $\Phi_{/\!\!/}$ and $\Phi_{/}$ denote the classes of left weights defined by the following:

$$\Phi_{/\!\!/} \ni \varphi' \overset{\text{def}}{\iff} \varphi \mathbin{/\!\!/} \varphi' \text{ for all } \varphi \in \Phi;$$

$$\Phi_{/} \ni \varphi' \overset{\text{def}}{\iff} \varphi \mathbin{/} \varphi' \text{ for all } \varphi \in \Phi.$$

Remark

$\Phi_{/\!\!/}$ and $\Phi_{/}$ become left dogmas.

Soundness

Definition (Adámek, Borceux, et al. 2002)

A class Φ of left weights is sound $\overset{\text{def}}{\Leftrightarrow}$ $\Phi_{/\!/} = \Phi_{/}$

Φ: sound $\rightsquigarrow$ Theory of $\Phi_{/\!/}(=\Phi_{/})$-accessible categories behaves well.

Example in $\mathbf{Set}$-$\mathbb{P}\mathrm{rof}$

A class $\mathrm{Fin} = \{$left weights of finite (co)limits$\}$ is sound.
Then, $\mathrm{Fin}_{/\!/} = \mathrm{Fin}_{/} = \{$l.w. of filtered colim$\}$.

Adjunctions of weights

Definition

A (horizontal) adjunction $(\psi \dashv \varphi)$ consists of:

- Horizontal arrows $Y \overset{\psi}{\nrightarrow} X$, $X \overset{\varphi}{\nrightarrow} Y$;
- Cells
$$\begin{tikzcd} & Y \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\eta"] & \\ Y \arrow[rr, "\psi\odot\varphi"', "|" marking] & {} & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi", "|" marking] \arrow[dr, equal] & Y \arrow[r, "\psi", "|" marking] \arrow[d, phantom, "\varepsilon"] & X \arrow[dl, equal] \\ & X & \end{tikzcd}$$
s.t. the *zigzag identities* hold.

Theorem

$\psi \dashv \varphi$: hor.adj. Then, ψ: a right weight $\Leftrightarrow$ φ: a left weight.

Definition

A left weight $X \overset{\varphi}{\nrightarrow} Y$ is left-absolute $\overset{\text{def}}{\Leftrightarrow}$ "φ-colimits are always absolute."

Theorem (Street 1983)

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$, $X \overset{\varphi}{\nrightarrow} Y$ has a left adjoint $\Leftrightarrow$ φ is left-absolute.

Street's characterization in a virtual equipment

Definition

$\mathbb{E}$ has anti-restrictions $\overset{\text{def}}{\Leftrightarrow}$ For every $X \overset{u}{\rightarrow\!\!\!\!\!+} Y$,
$$\begin{tikzcd} X \arrow[rr, "u", "\shortmid"{marking}] \arrow[dr, "\exists"'] & \arrow[d, phantom, "\exists\text{cart}"] & Y \arrow[dl, "\exists"] \\ & \exists Z & \end{tikzcd}$$

Theorem

$X \overset{\varphi}{\rightarrow\!\!\!\!\!+} Y$: a left weight

$$\begin{tikzcd} \exists\psi \dashv \varphi \arrow[dd, Rightarrow] & \\ & \varphi \,/\, \varphi \text{ (weakly comm.)} \arrow[ul, Rightarrow] \\ \varphi:\ \text{left-absolute} \arrow[ur, Rightarrow, "\text{if } \mathbb{E} \text{ has anti-res. and } \Phi^{\triangledown}X\ (\varphi \in \Phi) \text{ exists.}"'] & \end{tikzcd}$$

Characterization of ind-morphisms

Φ: a class of left weights.

Definition

$X \xrightarrow{f} Y$ is Φ-atomic $\overset{\text{def}}{\Leftrightarrow}$ $\forall A \overset{\forall\varphi}{\nrightarrow} \forall B$ in Φ, $A \overset{\forall g}{\to} Y$, $\exists \alpha, \beta$ s.t.

$$\begin{tikzcd} X \arrow[r, "Y(f{,}g)", "\shortmid"{marking}] \arrow[dr, "f"'] & A \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, "g"] \arrow[d, phantom, "\text{cart}"', shift right=3] \arrow[d, phantom, "\text{lan}", shift left=4] & B \arrow[dl] \\ & Y & \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, "Y(f{,}g)", "\shortmid"{marking}] \arrow[d, equal] & A \arrow[r, "\varphi", "\shortmid"{marking}] \arrow[d, phantom, "\alpha\colon \text{comp}"] & B \arrow[d, equal] \\ X \arrow[rr, "\shortmid"{marking}] \arrow[dr, "f"'] & \arrow[d, phantom, "\beta\colon \text{cart}"] & B \arrow[dl] \\ & Y & \end{tikzcd}$$

whenever lan exists.

In $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$,

$\mathbf{X} \xrightarrow{F} \mathbf{Y}$ is Φ-atomic $\Leftrightarrow$ $\forall x \in \mathbf{X}$, $\mathbf{Y}(Fx, -)\colon \mathbf{Y} \to \mathscr{V}$ is Φ-cocontinuous.

Characterization of ind-morphisms

Φ: a class of left weights.

Theorem

$A \xrightarrow{k} X$ is a Φ-ind-morphism. $\Leftrightarrow$

- X is Φ-cocomplete;
- k is Φ-atomic and *fully faithful*;
- For any $Y \xrightarrow{f} X$, there exist B, $B \overset{\varphi}{\nrightarrow} Y$ in Φ, $B \xrightarrow{g} A$, and a lan-cell:

$$\begin{tikzcd} B \arrow[r, "\varphi", "|"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\mathsf{lan}"] & Y \arrow[ddl, "f"] \\ A \arrow[d, "k"'] & \\ X & \end{tikzcd}$$

The 3rd condition says that "every $x \in X$ is a Φ-colimit of Φ-atomic objects."

The functors in detail

$$\begin{tikzcd} \mathbb{E}_\Phi{}' \arrow[r, phantom, "\subseteq"] \arrow[dr, "\Phi^\nabla"'] & \mathbb{E}_\Phi \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] \\ & \mathbb{Q}\Phi^\nabla \arrow[u, "U"'] \end{tikzcd}
\qquad
\begin{array}{c} A \\ \eta_A\downarrow \\ U\Phi^\nabla A \end{array} := \begin{array}{c} A \\ a\downarrow \\ \Phi^\nabla A \end{array} \text{ (a } \Phi\text{-ind-morphism in } \mathbb{E})$$

fullsub $\mathbb{E}_\Phi{}' := \{A \mid \Phi^\nabla A \text{ exists}\} \subseteq \mathbb{E}_\Phi$.

Definition of Φ^∇

- $A \quad \mapsto \quad \Phi^\nabla A$
- $\begin{array}{c} A \\ f\downarrow \\ B \end{array} \quad \mapsto \quad \begin{tikzcd} A \arrow[r, "a_*"] \arrow[d, "f"'] & \Phi^\nabla A \arrow[ddl, "\Phi^\nabla f"] \\ B \arrow[d, "b"'] \arrow[r, phantom, "\text{lan}"] & \\ \Phi^\nabla B & \end{tikzcd}$ in $\mathbb{E}$
- $A \overset{u}{\to} B \quad \mapsto \quad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a"'] & B \arrow[r, "b_*"] \arrow[d, phantom, "\text{lan}"] & \Phi^\nabla B \arrow[dll, "\Phi^\nabla u"] \\ \Phi^\nabla A & & \end{tikzcd}$ in $\mathbb{E}$

Definition of U

$$\begin{array}{ccc} X & \overset{u}{\leftarrow} & Y \\ f\downarrow & & \downarrow g \\ Z & \underset{v}{\leftarrow} & W \end{array} \quad \mapsto \quad \begin{array}{ccc} X & \overset{u^*}{\to} & Y \\ f\downarrow & & \downarrow g \\ Z & \underset{v^*}{\to} & W \end{array} \text{ in } \mathbb{E}$$

Relative companied biadjoints

We fix the following data:

- pseudo-double categories $\mathbb{A}'$, $\mathbb{A}$, and $\mathbb{B}$;
- "pseudo-double functors" $\mathbb{A}' \xrightarrow{I} \mathbb{A}$, $\mathbb{A}' \xrightarrow{F} \mathbb{B}$, and $\mathbb{B} \xrightarrow{G} \mathbb{A}$;
- a pseudo-vertical trans $I \overset{\eta}{\Rightarrow} GF$ whose components have companions.

$$
\begin{tikzcd}
\mathbb{A}' \arrow[r, "I"] \arrow[dr, "F"'] \arrow[dr, phantom, bend left, "\Searrow\eta"] & \mathbb{A} \\
& \mathbb{B} \arrow[u, "G"']
\end{tikzcd}
$$

(HTrans): The "horizontal naturality" of η

$$
\begin{pmatrix} F \\ G \end{pmatrix}:\;
\begin{array}{c} FA \\ f\downarrow \\ B \end{array}
\;\Bigg\|\;
\begin{array}{c} IA \\ \hat{f}\downarrow \\ GB \end{array}
\qquad
\begin{pmatrix} F & G \end{pmatrix}:\;
\frac{FA \overset{u}{\nrightarrow} B}{IA \overset{\hat{u}}{\nrightarrow} GB}
$$

$$
\begin{pmatrix} F & G \\ G & G \end{pmatrix}:\;
\begin{array}{ccc} FA & \overset{u}{\nrightarrow} & B_0 \\ f\downarrow & \alpha & \downarrow g \\ B_1 & \underset{v}{\nrightarrow} & B_2 \end{array}
\;\Bigg\|\;
\begin{array}{ccc} IA & \overset{\hat{u}}{\nrightarrow} & GB_0 \\ \hat{f}\downarrow & \hat{\alpha} & \downarrow Gg \\ GB_1 & \underset{Gv}{\nrightarrow} & GB_2 \end{array}
\qquad
\begin{pmatrix} F & F \\ F & G \end{pmatrix}:\;
\begin{array}{ccc} FA_0 & \overset{Fu}{\nrightarrow} & FA_1 \\ Ff\downarrow & \alpha & \downarrow g \\ FA_2 & \underset{v}{\nrightarrow} & B \end{array}
\;\Bigg\|\;
\begin{array}{ccc} IA_0 & \overset{Iu}{\nrightarrow} & IA_1 \\ If\downarrow & \hat{\alpha} & \downarrow \hat{g} \\ IA_2 & \underset{\hat{v}}{\nrightarrow} & GB \end{array}
$$

$$
\begin{pmatrix} F & G \\ F & G \end{pmatrix}:\;
\begin{array}{ccc} FA_0 & \overset{u}{\nrightarrow} & B_0 \\ Ff\downarrow & \alpha & \downarrow g \\ FA_1 & \underset{v}{\nrightarrow} & B_1 \end{array}
\;\Bigg\|\;
\begin{array}{ccc} IA_0 & \overset{\hat{u}}{\nrightarrow} & GB_0 \\ If\downarrow & \hat{\alpha} & \downarrow Gg \\ IA_1 & \underset{\hat{v}}{\nrightarrow} & GB_1 \end{array}
\qquad
\begin{pmatrix} F & F \\ G & G \end{pmatrix}:\;
\begin{array}{ccc} FA_0 & \overset{Fu}{\nrightarrow} & FA_1 \\ f\downarrow & \alpha & \downarrow g \\ B_0 & \underset{v}{\nrightarrow} & B_1 \end{array}
\;\Bigg\|\;
\begin{array}{ccc} IA_0 & \overset{Iu}{\nrightarrow} & IA_1 \\ \hat{f}\downarrow & \hat{\alpha} & \downarrow \hat{g} \\ GB_0 & \underset{Gv}{\nrightarrow} & GB_1 \end{array}
$$

Relationship among the 7 axioms

Under $\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} F & G \end{smallmatrix}\right)$, implications of the following directions hold.

$$\begin{tikzcd}
\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right) \arrow[rr] \arrow[dd] & & \left(\begin{smallmatrix} F & G \\ F & G \end{smallmatrix}\right) \arrow[dl, bend left] \\
& \text{(HTrans)} \arrow[dl, bend right] & \\
\left(\begin{smallmatrix} F & F \\ G & G \end{smallmatrix}\right) \arrow[rr] & & \left(\begin{smallmatrix} F & F \\ F & G \end{smallmatrix}\right) \arrow[ul, bend right]
\end{tikzcd}$$

Definition

$$\begin{tikzcd}
\mathbb{A}' \arrow[r, "I"] \arrow[dr, "F"'] & \mathbb{A} \arrow[d, phantom, "\Downarrow\eta"] \\
& \mathbb{B} \arrow[u, "G"']
\end{tikzcd}$$
forms an I-relative companied biadjunction $\overset{\text{def}}{\Leftrightarrow}$ $\left(\begin{smallmatrix} F \\ G \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} F & G \end{smallmatrix}\right)$, (HTrans), $\left(\begin{smallmatrix} F & G \\ G & G \end{smallmatrix}\right)$ hold.

$\Leftrightarrow$ All the 7 axioms hold.

Theorem

$$\begin{tikzcd}
\mathbb{E}_\Phi{}' \arrow[r, phantom, "\subseteq"] \arrow[dr, "\Phi^\nabla"'] & \mathbb{E}_\Phi \arrow[d, phantom, "\Downarrow\eta"] \\
& \mathbb{Q}\Phi^\nabla \arrow[u, "U"']
\end{tikzcd}$$
forms a relative companied biadjunction.

Nerves and realizations

Theorem (Companion theorem)

$$\begin{tikzcd} \mathbb{A}' \arrow[r, "I"] \arrow[dr, "F"'] & \mathbb{A} \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] \\ & \mathbb{B} \arrow[u, "G"'] \end{tikzcd} : \text{rel.comp-biadj} \quad \Longrightarrow$$

1. $\begin{matrix} FA \\ f\downarrow \\ B \end{matrix}$ has a companion $\Leftrightarrow$ $\begin{matrix} IA \\ \hat{f}\downarrow \\ GB \end{matrix}$ has a companion.
2. $FA \overset{u}{\relbar\joinrel\mapstochar\joinrel\rightarrow} B$ is a companion $\Leftrightarrow$ $IA \overset{\hat{u}}{\relbar\joinrel\mapstochar\joinrel\rightarrow} GB$ is a companion.

Corollary

1. $$\begin{tikzcd} A \arrow[r, "a_*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\text{lan}" near start] & \Phi^{\nabla}A \arrow[dl, "l"] \\ \underset{\Phi\text{-cocpl}}{\underline{E}} & \end{tikzcd}$$ Then, $f_* \in \Phi \quad \Leftrightarrow \quad l$ has a right adjoint.

2. $$\begin{tikzcd} A \arrow[r, "\varphi\in\Phi", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\text{lan}" near start] & \underset{\Phi\text{-cocpl}}{\underline{E}} \arrow[dl, "r"] \\ \Phi^{\nabla}A & \end{tikzcd}$$ Then, φ is a companion $\Leftrightarrow$ r has a left adjoint.