Double categories of profunctors

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← Today's slides

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Abstract

$\mathbb{X}$: a "double category" (*unital virtual double category*)

$\rightsquigarrow$

The "double category" $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$:

- $\mathbb{X}$-enriched categories;
- $\mathbb{X}$-enriched functors;
- $\mathbb{X}$-enriched profunctors.

Goal

To characterize the "double categories" $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$.

Enrichment in a monoidal category

$(\mathscr{V}, \otimes, I)$: a monoidal category

$\mathscr{V}$-category $\mathbf{A}$

- A class $\mathrm{Ob}\mathbf{A}$ of *objects*;
- *Hom-objects* $\mathbf{A}(x,y) \in \mathscr{V}$ $(x, y \in \mathrm{Ob}\mathbf{A})$;
- *Compositions* $\mathbf{A}(x,y) \otimes \mathbf{A}(y,z) \to \mathbf{A}(x,z)$ in $\mathscr{V}$ $(x, y, z \in \mathrm{Ob}\mathbf{A})$;
- *Identities* $I \to \mathbf{A}(x,x)$ in $\mathscr{V}$ $(x \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

$\mathscr{V}$-functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\begin{array}{ccc} \mathrm{Ob}\mathbf{A} & \longrightarrow & \mathrm{Ob}\mathbf{B}; \\ x & \longmapsto & Fx \end{array}$
- $\mathbf{A}(x,y) \to \mathbf{B}(Fx, Fy)$ in $\mathscr{V}$ $(x, y \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

$\mathscr{V}$-profunctor $\mathbf{A} \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!\!\!\;\;\;\;+} \mathbf{B}$

- *Hom-objects* $P(x,y) \in \mathscr{V}$ $(x \in \mathbf{A}, y \in \mathbf{B})$;
- *Actions* $\mathbf{A}(x',x) \otimes P(x,y) \to P(x',y)$, $P(x,y) \otimes \mathbf{B}(y,y') \to P(x,y')$ in $\mathscr{V}$.

(+Axioms)

The virtual double category $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

- (objects) $\mathscr{V}$-categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$;
- (vertical arrows) $\mathscr{V}$-functors $\begin{array}{c}\mathbf{A}\\ F\downarrow\\ \mathbf{B}\end{array}$, … and their compositions and identities;
- (horizontal arrows) $\mathscr{V}$-profunctors $\mathbf{A} \xrightarrow{P} \mathbf{B}$, …;
- (cells) "generalized" $\mathscr{V}$-natural transformations

$$\begin{array}{ccccccc} \mathbf{A}_0 & \overset{P_1}{\nrightarrow} & \mathbf{A}_1 & \overset{P_2}{\nrightarrow} & \cdots & \overset{P_n}{\nrightarrow} & \mathbf{A}_n \\ F\downarrow & & & \alpha & & & \downarrow G \\ \mathbf{B} & & & \xrightarrow[Q]{} & & & \mathbf{C} \end{array} \quad \Bigg\| \quad \begin{array}{c} P_1(x_0, x_1) \otimes P_2(x_1, x_2) \otimes \cdots \otimes P_n(x_{n-1}, x_n) \\ \downarrow \\ Q(Fx_0, Gx_n) \end{array}$$

- and their compositions (+ identity cells)

$$\begin{array}{ccccccc} \mathbf{A}_0 & \overset{\vec{P}_1}{\dashrightarrow} & \mathbf{A}_1 & \overset{\vec{P}_2}{\dashrightarrow} & \cdots & \overset{\vec{P}_n}{\dashrightarrow} & \mathbf{A}_n \\ F_0\downarrow & \alpha_1 & F_1\downarrow & \alpha_2 & \cdots & \alpha_n & \downarrow F_n \\ \mathbf{B}_0 & \xrightarrow[Q_1]{} & \mathbf{B}_1 & \xrightarrow[Q_2]{} & \cdots & \xrightarrow[Q_n]{} & \mathbf{B}_n \\ G\downarrow & & & \beta & & & \downarrow H \\ \mathbf{C} & & & \xrightarrow[R]{} & & & \mathbf{D} \end{array} \rightsquigarrow \begin{array}{ccccccc} \mathbf{A}_0 & \overset{\vec{P}_1}{\dashrightarrow} & \mathbf{A}_1 & \overset{\vec{P}_2}{\dashrightarrow} & \cdots & \overset{\vec{P}_n}{\dashrightarrow} & \mathbf{A}_n \\ F_0 ⨟ G\downarrow & & & \vec{\alpha} ⨟ \beta & & & \downarrow F_n ⨟ H \\ \mathbf{C} & & & \xrightarrow[R]{} & & & \mathbf{D} \end{array}$$

Horizontal composition

$\mathbb{X}$: a virtual double category (VDC).

Definition

A cell
$$\begin{array}{ccccccc} A_0 & \overset{u_1}{\nrightarrow} & A_1 & \overset{u_2}{\nrightarrow} & \cdots & \overset{u_n}{\nrightarrow} & A_n \\ \| & & & \alpha & & & \| \\ A_0 & & & \underset{v}{\nrightarrow} & & & A_n \end{array}$$
in $\mathbb{X}$ is composing

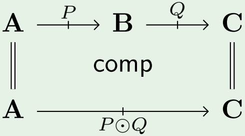
$$\overset{\text{def}}{\Leftrightarrow} \quad \begin{array}{ccccccc} \cdot & \dashrightarrow & A_0 & \overset{\vec{u}}{\dashrightarrow} & A_n & \dashrightarrow & \cdot \\ \downarrow & & & \forall\beta & & & \downarrow \\ \cdot & & & \nrightarrow & & & \cdot \end{array} \quad = \quad \begin{array}{ccccccc} \cdot & \dashrightarrow & A_0 & \overset{\vec{u}}{\dashrightarrow} & A_n & \dashrightarrow & \cdot \\ \| & \text{॥} & \| & \alpha & \| & \text{॥} & \| \\ \cdot & \dashrightarrow & A_0 & \underset{v}{\nrightarrow} & A_n & \dashrightarrow & \cdot \\ \downarrow & & & \exists!\gamma & & & \downarrow \\ \cdot & & & \nrightarrow & & & \cdot \end{array}$$
in $\mathbb{X}$.

Definition

The unit on $X \in \mathbb{X}$ $\cdots$
$$\begin{array}{ccc} & X & \\ /\!/ & \text{comp} & \backslash\!\backslash \\ X & \underset{\mathsf{U}_X}{\nrightarrow} & X \end{array}$$
in $\mathbb{X}$. (written
$$\begin{array}{ccc} & X & \\ /\!/ & \text{comp} & \backslash\!\backslash \\ X & \overset{}{=\!=\!\!\!\!+\!\!\!\!=\!=} & X \end{array}$$
)

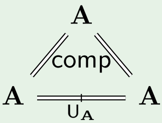
Horizontal composition

Examples in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

1. 

$$(P \odot Q)(x,z) := \int^{y \in \mathbf{B}} P(x,y) \otimes Q(y,z) \quad \text{in } \mathscr{V}$$

(Suppose that $\int$ is preserved by $X \otimes -, - \otimes Y$.)

2. 

$$\mathsf{U}_{\mathbf{A}}(x,x') := \mathbf{A}(x,x') \quad \text{in } \mathscr{V}$$

Unital virtual double categories

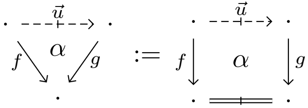
Definition

A VDC $\mathbb{X}$ is unital $\overset{\text{def}}{\Leftrightarrow}$ Every object $X \in \mathbb{X}$ has the unit.

Example

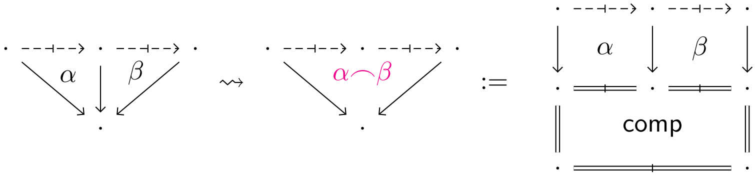
$\mathscr{V}$-$\mathbb{P}\mathrm{rof}$ is unital.

In a unital VDC $\mathbb{X}$,



in $\mathbb{X}$. (0-coary cell)

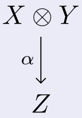
0-coary cells can be composed horizontally:



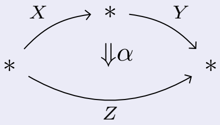
Monoidal categories vs. bicategories vs. VDCs

Monoidal category = single-object bicategory

$\mathscr{V}$: a mon.cat. $\quad\leftrightsquigarrow\quad$ $\mathcal{B}(\mathscr{V})$: a single-obj.bicat.



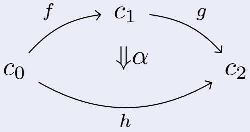
in $\mathscr{V}$



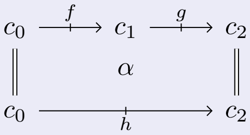
in $\mathcal{B}(\mathscr{V})$

(Virtual) bicategory = vertically discrete VDC

$\mathscr{W}$: a (virtual) bicat. $\quad\leftrightsquigarrow\quad$ $\mathbb{D}(\mathscr{W})$: a vertically discrete VDC



in $\mathscr{W}$



in $\mathbb{D}(\mathscr{W})$

Generalization of enriching bases

Enrichment in...

- VDC $\mathbb{X}$
- $\cup\!\wr$ (generalize)
- bicategory $\mathscr{W}$
- $\cup\!\wr$ (generalize)
- mon.category $\mathscr{V}$

Remark

We obtain unital VDCs $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$, $\mathscr{W}$-$\mathbb{P}\mathrm{rof}$, and $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$ for any $\mathscr{V}, \mathscr{W}, \mathbb{X}$.

From $\mathscr{V}$ to $\mathscr{W}$

$\mathscr{W}$: a bicategory

$\mathscr{W}$-category $\mathbf{A}$

- A class $\mathrm{Ob}\mathbf{A}$ of *objects*;
- *Coloring* $|x| \in \mathscr{W}$ $(x \in \mathrm{Ob}\mathbf{A})$;
- *Hom-1-cells* $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in $\mathscr{W}$ $(x, y \in \mathrm{Ob}\mathbf{A})$;
- *Compositions* $$\begin{array}{c} |x| \xrightarrow{\mathbf{A}(x,y)} |y| \xrightarrow{\mathbf{A}(y,z)} |z| \\ \Downarrow \\ |x| \xrightarrow[\mathbf{A}(x,z)]{} |z| \end{array}$$ in $\mathscr{W}$ $(x, y, z \in \mathrm{Ob}\mathbf{A})$;
- *Identities* $$\begin{array}{c} |x| \overset{1_{|x|}}{=\!=\!=} |x| \\ \Downarrow \\ |x| \xrightarrow[\mathbf{A}(x,x)]{} |x| \end{array}$$ in $\mathscr{W}$ $(x \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

From $\mathscr{V}$ to $\mathscr{W}$

$\mathscr{W}$: a bicategory

$\mathscr{W}$-functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\mathrm{Ob}\mathbf{A} \longrightarrow \mathrm{Ob}\mathbf{B}$ s.t. $|x| = |Fx|$ in $\mathscr{W}$;
  $x \longmapsto Fx$
- $|x| \underset{\mathbf{B}(Fx,Fy)}{\overset{\mathbf{A}(x,y)}{\Downarrow}} |y|$ in $\mathscr{W}$ $(x, y \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

$\mathscr{W}$-profunctor $\mathbf{A} \overset{P}{\relbar\!\!\!\mapsto} \mathbf{B}$

- *Hom-1-cells* $|x| \xrightarrow{P(x,y)} |y|$ in $\mathscr{W}$ $(x \in \mathbf{A}, y \in \mathbf{B})$;
- *Actions* $|x'| \xrightarrow{\mathbf{A}(x',x)} |x| \xrightarrow{P(x,y)} |y| \ \Downarrow\ |x'| \xrightarrow{P(x',y)} |y|$ and $|x| \xrightarrow{P(x,y)} |y| \xrightarrow{\mathbf{B}(y,y')} |y'| \ \Downarrow\ |x| \xrightarrow{P(x,y')} |y'|$ in $\mathscr{W}$.

(+Axioms)

From $\mathscr{V}$ to $\mathscr{W}$

Enrichment in... $\left\{ \begin{array}{c} \text{VDC } \mathbb{X} \\ \rotatebox{90}{$\subsetneq$} \\ \text{bicategory } \mathscr{W} \\ \rotatebox{90}{$\subsetneq$} \\ \text{mon.category } \mathscr{V} \end{array} \right.$

Objects are colored with $\mathrm{Ob}\mathscr{W}$.

From $\mathscr{W}$ to $\mathbb{X}$

$\mathbb{X}$: a VDC

$\mathbb{X}$-category $\mathbf{A}$

- A class $\mathrm{Ob}\mathbf{A}$ of *objects*;
- *Coloring* $|x| \in \mathbb{X}$ $(x \in \mathrm{Ob}\mathbf{A})$;
- *Hom-horizontal arrows* $|x| \xrightarrow{\mathbf{A}(x,y)} |y|$ in $\mathbb{X}$ $(x, y \in \mathrm{Ob}\mathbf{A})$;
- *Compositions* $\begin{array}{ccccc} |x| & \xrightarrow{\mathbf{A}(x,y)} & |y| & \xrightarrow{\mathbf{A}(y,z)} & |z| \\ \| & & \bullet & & \| \\ |x| & & \xrightarrow[\mathbf{A}(x,z)]{} & & |z| \end{array}$ in $\mathbb{X}$ $(x, y, z \in \mathrm{Ob}\mathbf{A})$;
- *Identities* $\begin{array}{ccc} & |x| & \\ // & \bullet & \backslash\backslash \\ |x| & \xrightarrow[\mathbf{A}(x,x)]{} & |x| \end{array}$ in $\mathbb{X}$ $(x \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

From $\mathscr{W}$ to $\mathbb{X}$

$\mathbb{X}$: a VDC

Notation

$\mathcal{H}(\mathbb{X})$ $\cdots$ the (virtual) bicat. obtained by forgetting all vertical arrows from $\mathbb{X}$.

Enrichment in a bicategory		Enrichment in a VDC
$\mathcal{H}(\mathbb{X})$-categories	$=$	$\mathbb{X}$-categories
$\mathcal{H}(\mathbb{X})$-functors	$\subseteq$	$\mathbb{X}$-functors
$\mathcal{H}(\mathbb{X})$-profunctors	$=$	$\mathbb{X}$-profunctors

From $\mathscr{W}$ to $\mathbb{X}$

$\mathbb{X}$-functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$

- A map $\mathrm{Ob}\mathbf{A} \longrightarrow \mathrm{Ob}\mathbf{B}$;
  $x \longmapsto F^0x$
- "*Color-comparing*" vertical arrows $|x| \xrightarrow{F^1x} |F^0x|$ in $\mathbb{X}$ $(x \in \mathrm{Ob}\mathbf{A})$;
- $$\begin{array}{ccc} |x| & \xrightarrow{\mathbf{A}(x,y)} & |y| \\ {\scriptstyle F^1x}\downarrow & \bullet & \downarrow{\scriptstyle F^1y} \\ |F^0x| & \xrightarrow[\mathbf{B}(F^0x,F^0y)]{} & |F^0y| \end{array}$$ in $\mathbb{X}$ $(x, y \in \mathrm{Ob}\mathbf{A})$. (+Axioms)

Enrichment in... { VDC $\mathbb{X}$ $\cup\!\!\!\nmid$ bicategory $\mathscr{W}$ $\cup\!\!\!\nmid$ mon.category $\mathscr{V}$ }

- Between VDC $\mathbb{X}$ and bicategory $\mathscr{W}$: Functors no longer need to preserve the color of objects.
- Between bicategory $\mathscr{W}$ and mon.category $\mathscr{V}$: Objects are colored with $\mathrm{Ob}\mathscr{W}$.

Collages of a profunctor

$\mathscr{V}$: a monoidal category with $\varnothing$ (the initial preserved by $\otimes$)

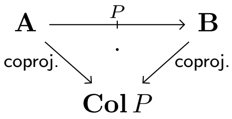
$\mathbf{A} \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbf{B}$: a $\mathscr{V}$-profunctor

Definition

A collage (or *cograph*) of P is the $\mathscr{V}$-category $\mathbf{Col}\,P$:

$$\mathrm{Ob}(\mathbf{Col}\,P) := \mathrm{Ob}\mathbf{A} + \mathrm{Ob}\mathbf{B};$$

$$\mathbf{Col}\,P(x,y) := \begin{cases} \mathbf{A}(x,y) & \text{if } x,y \in \mathbf{A} \\ \mathbf{B}(x,y) & \text{if } x,y \in \mathbf{B} \\ P(x,y) & \text{if } x \in \mathbf{A}, y \in \mathbf{B} \\ \varnothing & \text{if } x \in \mathbf{B}, y \in \mathbf{A} \end{cases}$$



in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

Universal property (vertical)

$$\begin{array}{c}\mathbf{Col}\,P\\ \downarrow\\ \mathbf{C}\end{array} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof} \qquad \Bigg\| \qquad \begin{array}{ccc}\mathbf{A} & \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\;} & \mathbf{B}\\ & \searrow \;\bullet\; \swarrow & \\ & \mathbf{C} & \end{array} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$$

Universal property (horizontal)

$$\mathbf{Col}\,P \longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\; \mathbf{C} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof} \qquad \Bigg\| \qquad \begin{array}{ccccc}\mathbf{A} & \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\;} & \mathbf{B} & \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\; & \mathbf{C}\\ \| & & \bullet & & \|\\ \mathbf{A} & & \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\; & & \mathbf{C}\end{array} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$$

$$\mathbf{C} \longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\; \mathbf{Col}\,P \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof} \qquad \Bigg\| \qquad \begin{array}{ccccc}\mathbf{C} & \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\; & \mathbf{A} & \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\;} & \mathbf{B}\\ \| & & \bullet & & \|\\ \mathbf{C} & & \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\; & & \mathbf{B}\end{array} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof}$$

$\mathbf{Col}\,P$ has universal properties in the three directions: $\begin{array}{c}\cdot \;\longrightarrow\!\!\!\!\!\!+\;\; \mathbf{Col}\,P \;\longrightarrow\!\!\!\!\!\!+\;\; \cdot\\ \downarrow\\ \cdot\end{array}$

Collages in general

S: a set

Notation

$\mathbb{I}\mathrm{S}$ $\cdots$ the VDC described by:

- $\mathrm{Ob}(\mathbb{I}\mathrm{S}) := \mathrm{S}$;
- For $i, j \in \mathrm{S}$, there is a unique horizontal arrow $i \xrightarrow{!_{ij}} j$ in $\mathbb{I}\mathrm{S}$;
- For $i_0, \ldots, i_n \in \mathrm{S}$, there is a unique cell

$$\begin{array}{ccccccccc} i_0 & \xrightarrow{!_{i_0 i_1}} & i_1 & \xrightarrow{!_{i_1 i_2}} & \cdots & \xrightarrow{!_{i_{n-1} i_n}} & i_n \\ \| & & & !_{i_0 \ldots i_n} & & & \| \\ i_0 & & & \xrightarrow[!_{i_0 i_n}]{} & & & i_n \end{array}$$

A virtual double (VD)-functor $\mathbb{I}\mathrm{S} \longrightarrow \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ is equivalent to the following data:

- $\mathbf{A}_i$: $\mathbb{X}$-categories $(i \in \mathrm{S})$
- $\mathbf{A}_i \xrightarrow{P_{ij}} \mathbf{A}_j$: $\mathbb{X}$-profunctors $(i, j \in \mathrm{S})$
- $$\begin{array}{ccc} & \mathbf{A}_i & \\ // & \eta_i & \backslash\backslash \\ \mathbf{A}_i & \xrightarrow[P_{ii}]{} & \mathbf{A}_i \end{array} \qquad (i \in \mathrm{S})$$
- $$\begin{array}{ccccc} \mathbf{A}_i & \xrightarrow{P_{ij}} & \mathbf{A}_j & \xrightarrow{P_{jk}} & \mathbf{A}_k \\ \| & & \mu_{ijk} & & \| \\ \mathbf{A}_i & & \xrightarrow[P_{ik}]{} & & \mathbf{A}_k \end{array} \qquad (i, j, k \in \mathrm{S})$$

(+Axioms)

Collages in general

$P := (\mathbf{A}_i, P_{ij}, \eta_i, \mu_{ijk})_{i,j,k \in \mathrm{S}}$: a VD-functor $\mathbb{I}\mathrm{S} \xrightarrow{P} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$

Definition

A collage of P is the $\mathbb{X}$-category $\mathbf{Col}\, P$:

$$\mathrm{Ob}(\mathbf{Col}\, P) := \coprod_{i \in \mathrm{S}} \mathrm{Ob}\mathbf{A}_i;$$

$$\mathbf{Col}\, P(x, y) := P_{ij}(x, y) \quad \text{(where } x \in \mathbf{A}_i, y \in \mathbf{A}_j\text{)}.$$

$$\begin{array}{ccc} \mathbf{A}_i & \xrightarrow[\quad \xi_{ij} \quad]{P_{ij}} & \mathbf{A}_j \\ & {}_{\xi_i}\searrow \quad \swarrow_{\xi_j} & \\ & \mathbf{Col}\, P & \end{array}$$

in $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ (*coprojections*)

Again, $\mathbf{Col}\, P$ has universal properties in the three directions:

$$\begin{array}{ccccc} \cdot & \to & \mathbf{Col}\, P & \to & \cdot \\ & & \downarrow & & \\ & & \cdot & & \end{array}$$

Versatile colimits

$\mathbb{K}$: a VDC, $\mathbb{L}$: a unital VDC, $\mathbb{K} \xrightarrow{F} \mathbb{L}$: a VD-functor

Definition (informal)

A versatile colimit of F is a "cocone"

$$\left\{ \begin{array}{ccc} FA & \xrightarrow[\ \ \xi_u\ \ ]{Fu} & FB \\ & {\scriptstyle \xi_A} \searrow \quad \swarrow {\scriptstyle \xi_B} & \\ & \Xi & \end{array} \quad \text{in } \mathbb{L} \right\}_{\left(A \xrightarrow{u} B \quad \text{in } \mathbb{K}\right)}$$

having the universal property in the three directions $\begin{array}{ccccc} \cdot & \to & \Xi & \to & \cdot \\ & & \downarrow & & \\ & & \cdot & & \end{array}$ in $\mathbb{L}$.

Example

1. Versatile collages (:= vers.colim. of shapes $\mathbb{K} = \mathbb{IS}$)
2. Versatile coproducts (:= vers.colim. of discrete shapes)

The object classifier in $\mathscr{V}$-$\mathbb{P}\mathrm{rof}$

$(\mathscr{V}, \otimes, I)$: a monoidal category

Notation

The *unit $\mathscr{V}$-category* $\mathbf{I}$:

$$\mathrm{Ob}\mathbf{I} := \{*\}, \quad \mathbf{I}(*,*) := I.$$

The unit $\mathscr{V}$-category classifies objects

For every $\mathscr{V}$-category $\mathbf{A}$,

$$\begin{array}{c} \mathbf{I} \\ \downarrow \\ \mathbf{A} \end{array} \quad \text{in } \mathscr{V}\text{-}\mathbb{P}\mathrm{rof} \quad \Bigg\| \quad \text{an object } x \in \mathbf{A}$$

Object classifiers in $\mathscr{W}$-$\mathbb{P}\mathrm{rof}$

$\mathscr{W}$: a bicategory

Notation

The *unit $\mathscr{W}$-category* $\mathbf{I}_c$ for $c \in \mathscr{W}$:

$$\mathrm{Ob}\mathbf{I}_c := \{*\} \text{ with } |\,*\,| := c, \quad \mathbf{I}_c(*,*) := (c \xrightarrow{\mathrm{id}} c \quad \text{in } \mathscr{W})$$

The unit $\mathscr{W}$-categories classify objects

For every $\mathscr{W}$-category $\mathbf{A}$,

$$\frac{\mathbf{I}_c \to \mathbf{A} \quad \text{in } \mathscr{W}\text{-}\mathbb{P}\mathrm{rof}}{\text{an object } x \in \mathbf{A} \text{ s.t. } |x| = c}$$

Object classifiers in $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$

$\mathbb{X}$: a unital VDC

Notation

The *unit* $\mathbb{X}$*-category* $\mathbf{I}_c$ for $c \in \mathbb{X}$:

$$\mathrm{Ob}\mathbf{I}_c := \{*\} \text{ with } |\,*\,| := c, \quad \mathbf{I}_c(*,*) := (c \nrightarrow c \quad \text{in } \mathbb{X})$$

The unit $\mathbb{X}$-categories no longer classify objects.

For every $\mathbb{X}$-category $\mathbf{A}$,

$$\frac{\mathbf{I}_c \to \mathbf{A} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}}{\text{an object } x \in \mathbf{A}, \text{ and a vertical arrow } c \to |x| \text{ in } \mathbb{X}} \quad (\textit{semi-object})$$

$\rightsquigarrow$ The unit $\mathbb{X}$-categories classify *semi-objects*.

The embedding

$\mathbb{X}$: a unital VDC

Proposition

There is an "embedding" $\mathbb{X} \xhookrightarrow{\mathbf{I}_\bullet} \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$. That is:

$$\begin{array}{ccccccc} c_0 & \overset{}{\longrightarrow\!\!\!\!\!\!+} & c_1 & \longrightarrow\!\!\!\!\!\!+ & \cdots & \longrightarrow\!\!\!\!\!\!+ & c_n \\ \downarrow & & & \bullet & & & \downarrow \\ d & & & \longrightarrow\!\!\!\!\!\!+ & & & e \end{array} \quad \text{in } \mathbb{X} \quad \Bigg\| \quad \begin{array}{ccccccc} \mathbf{I}_{c_0} & \longrightarrow\!\!\!\!\!\!+ & \mathbf{I}_{c_1} & \longrightarrow\!\!\!\!\!\!+ & \cdots & \longrightarrow\!\!\!\!\!\!+ & \mathbf{I}_{c_n} \\ \downarrow & & & \bullet & & & \downarrow \\ \mathbf{I}_d & & & \longrightarrow\!\!\!\!\!\!+ & & & \mathbf{I}_e \end{array} \quad \text{in } \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$$

In what follows, we will consider $\mathbb{X} \subseteq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$.

Toward the characterization: density

$\mathbb{X} \subseteq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ ($\mathbb{X}$: a unital VDC)

Observation I

Every $\mathbb{X}$-category is a versatile collage of objects from $\mathbb{X}$.

$\because$

$$\frac{\mathbf{A} : \text{an } \mathbb{X}\text{-category}}{\mathbb{I}\mathrm{Ob}\mathbf{A} \xrightarrow{|\cdot|_{\mathbf{A}}} \mathbb{X} \subseteq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof} : \text{a VD-functor}}$$

⇣ ⇡ Taking the versatile colimit in $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$

The VD-functor $|\cdot|_{\mathbf{A}}$ is given by:

$$\begin{array}{ccccccc} x_0 & \overset{!}{\nrightarrow} & x_1 & \overset{!}{\nrightarrow} & \cdots & \overset{!}{\nrightarrow} & x_n \\ \| & & & ! & & & \| \\ x_0 & & & \underset{!}{\longrightarrow} & & & x_n \end{array} \quad \text{in } \mathbb{I}\mathrm{Ob}\mathbf{A}$$

$$\mapsto$$

$$\begin{array}{ccccccc} |x_0| & \xrightarrow{\mathbf{A}(x_0,x_1)} & |x_1| & \xrightarrow{\mathbf{A}(x_1,x_2)} & \cdots & \xrightarrow{\mathbf{A}(x_{n-1},x_n)} & |x_n| \\ \| & & & \text{“composition” in } \mathbf{A} & & & \| \\ |x_0| & & & \underset{\mathbf{A}(x_0,x_n)}{\longrightarrow} & & & |x_n| \end{array} \quad \text{in } \mathbb{X}$$

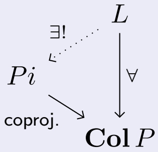
Toward the characterization: atomicity

Observation II

In $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$, the unit $\mathbb{X}$-categories can be characterized by "atomicity" w.r.t. versatile collages.

Definition

$\mathbb{L}$: a unital VDC.
$L \in \mathbb{L}$: collage-atomic $\overset{\text{def}}{\Leftrightarrow}$ For every (large) versatile collage $\mathbf{Col}\, P$ in $\mathbb{L}$, vertical arrows $L \to \mathbf{Col}\, P$ uniquely factor through a unique coprojection.



in $\mathbb{L}$ $\quad (\exists! i)$

Proposition

$\mathbb{X}$: a unital VDC.
In $\mathbb{X}$-$\mathbb{P}\mathrm{rof}$, collage-atomic $\iff$ vertically isomorphic to some $\mathbf{I}_c$.

The characterization

Theorem

TFAE for a unital VDC $\mathbb{L}$:

1. There is an equivalence $\mathbb{L} \simeq \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ for some unital VDC $\mathbb{X}$.
   (in the 2-category of unital VDCs)
2. - ◇ $\mathbb{L}$ has (large) versatile collages;
   - ◇ Every object in $\mathbb{L}$ is a versatile collage of collage-atomic objects.

Sketch of proof: The only non-trivial part is 2. $\Longrightarrow$ 1.
Let $\mathbb{X} := \{\text{collage-atomic objs.}\} \subseteq \mathbb{L}$.

We can construct an adjunction: $\mathbb{X}\text{-}\mathbb{P}\mathrm{rof} \underset{N}{\overset{R}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbb{L}$.

Construction of R: Regarding $\mathbf{A} \in \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$ as $\mathbb{I}\mathrm{Ob}\mathbf{A} \longrightarrow \mathbb{X} \subseteq \mathbb{L}$, we take the vers.colim. of it.

Construction of N: Suppose $L \in \mathbb{L}$ is a vers.colim. of $\mathbb{I}\mathrm{S} \xrightarrow{P} \mathbb{X} \subseteq \mathbb{L}$. Define $L \overset{N}{\longmapsto}$ "(S, P)" $\in \mathbb{X}\text{-}\mathbb{P}\mathrm{rof}$.

$R \circ N \cong \mathrm{Id}$ is trivial. $\mathrm{Id} \cong N \circ R$ follows from uniqueness of (S, P). □

Why is $\mathbb{IS} \xrightarrow{P} \mathbb{X}$ unique for each $L \in \mathbb{L}$?

- The uniqueness of the shape S and the object function of P
  $\cdots$ From the collage-atomicity of $\mathbb{X}$.
- The uniqueness of the horizontal arrow function & the cell function of P
  $\cdots$ From strongness theorem on versatile collages:

Strongness theorem

$\mathbb{L}$: a unital VDC. $\mathbf{Col}\,P(\in \mathbb{L})$: a versatile collage of $\mathbb{IS} \xrightarrow{P} \mathbb{L}$.
$\Longrightarrow$ Coprojections of $\mathbf{Col}\,P$ form *cartesian cells*:

$$\begin{array}{ccc} Pi & \xrightarrow{P!_{ij}} & Pj \\ & \text{cart} & \\ \text{coproj.} \searrow & & \swarrow \text{coproj.} \\ & \mathbf{Col}\,P & \end{array} \quad \text{in } \mathbb{L} \quad (i, j \in \mathrm{S}).$$

Thank you!

Today's slides

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