On the iterated analogue of the fundamental homomorphism theorem

Yuto Kawase (joint with Hayato Nasu)

RIMS, Kyoto University

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$\leftsquigarrow$ slides

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Recall

Every homomorphism $f\colon A \to B$ between groups can be decomposed into a surjective hom. followed by an injective hom.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & \operatorname{Im} f \arrow[ur, hook] & \end{tikzcd} \qquad \text{in } \mathbf{Grp}$$

Regular epimorphisms = morphisms being the coequalizer of some parallel pair of morphisms.

$$\cdot \rightrightarrows \cdot \xrightarrow{\text{regular epi}} \cdot$$

(eg. regular epi = surjective hom. in $\mathbf{Grp}$)

Monomorphisms = morphisms s.t. $f \circ -$ is injective.

$$\cdot \overset{g}{\underset{h}{\rightrightarrows}} \cdot \overset{\text{mono}}{\underset{f}{\hookrightarrow}} \cdot \quad \text{(cofork)} \qquad \Longrightarrow \qquad g = h$$

(eg. mono = injective hom. in $\mathbf{Grp}$)

Definition

$$f \perp g$$
(f and g are orthogonal)
$$\overset{\text{def}}{\Longleftrightarrow}$$
$$\begin{tikzcd}
\cdot \arrow[r, "\forall"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "\forall"] \arrow[ur, "\exists!"] & \cdot
\end{tikzcd}$$

Example

In every category, $\{\text{regular epi}\} \perp \{\text{mono}\}$.

$\mathbf{E} \subseteq \mathrm{Mor}\mathscr{C}$: a class of morphisms in a category $\mathscr{C}$.

Definition

$\mathbf{E}$: iso-closed $\overset{\text{def}}{\Leftrightarrow}$

- $\{\text{iso}\} \subseteq \mathbf{E}$.
- $$\begin{array}{ccc} \cdot & \cong & \cdot \\ \mathbf{E}\ni f\downarrow & & \downarrow g \\ \cdot & \cong & \cdot \end{array} \implies g \in \mathbf{E}.$$

Definition

$\mathbf{E}$: iso-closed.
$\mathscr{C}$ admits orthogonal $\mathbf{E}$-factorizations $\overset{\text{def}}{\Leftrightarrow}$ Every morphism f can be decomposed as

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "\mathbf{E}\ni e"'] & & \cdot \\ & \cdot \arrow[ur, "m\in\mathbf{r}(\mathbf{E})"'] & \end{tikzcd}$$

where $\mathbf{r}(\mathbf{E}) := \{m \mid \mathbf{E} \perp m\}$.

Example

$\mathbf{Grp}$ admits orthogonal $\{\text{regular epi}\}$-factorizations.

The fundamental homomorphism theorem

$$A/\operatorname{Ker} f \cong \operatorname{Im} f \qquad (f\colon A \to B)$$

The kernel pair of f:

$$\begin{tikzcd} \operatorname{Kp} f \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \text{(pullback)}$$

The coimage of f:

$$\begin{tikzcd} \operatorname{Kp} f \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\pi_1"'] & A \arrow[r, two heads] & \operatorname{Coim} f \end{tikzcd} \qquad \text{(coequalizer)}$$

The fundamental homomorphism theorem (categorically)

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & \operatorname{Coim} f \arrow[ur, hook, "\text{mono}"'] & \end{tikzcd}$$

This holds in any categories of equational algebras.

Fact

TFAE for a category $\mathscr{C}$ with finite limits and colimits:

1. $\mathscr{C}$ admits orthogonal $\{$regular epi$\}$-factorizations.
2. "The fundamental homomorphism theorem holds in $\mathscr{C}$." That is, for every $f\colon A \to B$ in $\mathscr{C}$, the canonical morphism $\operatorname{Coim} f \to B$ is monic.

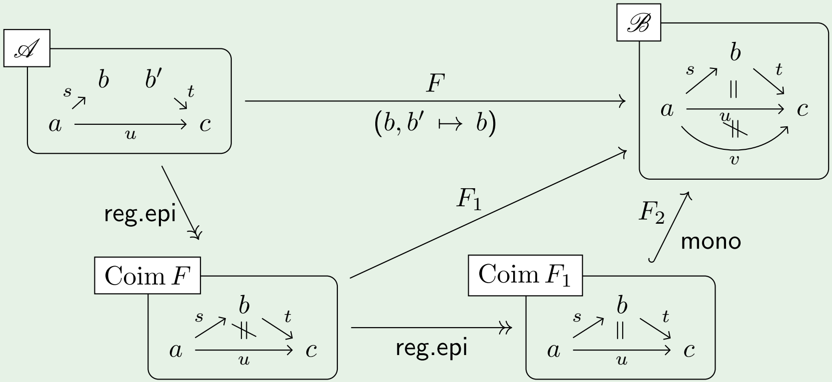
Question

Does the fundamental hom. thm. hold in any category?

$\rightsquigarrow$ No.

Counter example

In $\mathbf{Cat}$,



Definition

A (regular) decomposition of $A \xrightarrow{f} B$ in $\mathscr{C}$ consists of:

- A functor $\mathbf{Ord} \xrightarrow{A_\bullet} \mathscr{C}$ with $A_0 = A$;
- A cocone $f_\bullet = (A_\alpha \xrightarrow{f_\alpha} B)$ over $A_\bullet$ with $f_0 = f$

satisfying ($\forall \alpha$, $A_{\alpha,\alpha+1}$: regular epi) and ($\forall \gamma$: limit type, $A_\gamma \cong \mathrm{Colim}_{\alpha<\gamma} A_\alpha$).

$$
\begin{tikzcd}
A = A_0 \arrow[r, two heads, "A_{0,1}"] \arrow[d, "f=f_0"'] & A_1 \arrow[r] \arrow[dl, "f_1"'] & \cdots \arrow[r] & A_\alpha \arrow[r, two heads, "A_{\alpha,\alpha+1}"] \arrow[dlll, "f_\alpha"'] & A_{\alpha+1} \arrow[r] \arrow[dllll, bend left, "f_{\alpha+1}"] & \cdots \\
B
\end{tikzcd}
$$

in $\mathscr{C}$

Definition

1. A decomp. $(A_\bullet, f_\bullet)$ stabilizes at α $\overset{\text{def}}{\Leftrightarrow}$ f_β: mono ($\alpha \leq \forall \beta$) ($\Leftrightarrow$ f_α: mono)
2. The smallest α above is called the length of $(A_\bullet, f_\bullet)$. (if exists)

$\mathscr{C}$: a category with kernel pairs and colimits.

$$\begin{tikzcd}
A \arrow[r, two heads, "q_{0,1}"] \arrow[d, "f"'] & A_1 \arrow[r, two heads, "q_{1,2}"] \arrow[dl, "f_1"] & A_2 \arrow[dll, "f_2"] & \cdots & A_\omega \arrow[r, two heads, "q_{\omega,\omega+1}"] \arrow[dllll, "f_\omega"] & A_{\omega+1} \arrow[dlllll, "f_{\omega+1}"] & \cdots \\
B
\end{tikzcd} \quad \text{in } \mathscr{C}$$

A_1 : the coimage of f

A_2 : the coimage of f_1

A_ω : the colimit of the chain $(A_n)_{n<\omega}$

$A_{\omega+1}$: the coimage of f_ω

$\rightsquigarrow$ This gives a decomposition of f. (canonical decomposition)

Definition

1. The canonical decomposition number $\sigma(f)$:= the length of the canonical decomposition of f. If it doesn't stabilizes, $\sigma(f)$ is left to be undefined.
2. If $\sigma(f)$ is defined for every f in $\mathscr{C}$, the global canonical decomposition number $\sigma(\mathscr{C}) := \min\{\alpha \mid \sigma(f) < \alpha$ for every f in $\mathscr{C}\}$

Remark

1. $\sigma(\mathscr{C}) = 0 \iff \mathscr{C} = \varnothing$.
2. $\sigma(\mathscr{C}) \leq 1 \iff \sigma(f) = 0\ (\forall f) \iff f$: monic $(\forall f)$.
3. $\sigma(\mathscr{C}) \leq 2 \iff \sigma(f) \leq 1\ (\forall f) \iff$ "The fund. hom. thm. holds in $\mathscr{C}$."
4. $\sigma(\mathscr{C}) \leq \omega \iff \sigma(f)$: finite $(\forall f)$.

Example

1. $\sigma(\mathbf{Set}) = \sigma(\mathbf{Grp}) = 2$.
2. $\sigma(\mathbf{Cat}) = 3$.
3. $\sigma(n\text{-}\mathbf{Cat}) = n + 2$.

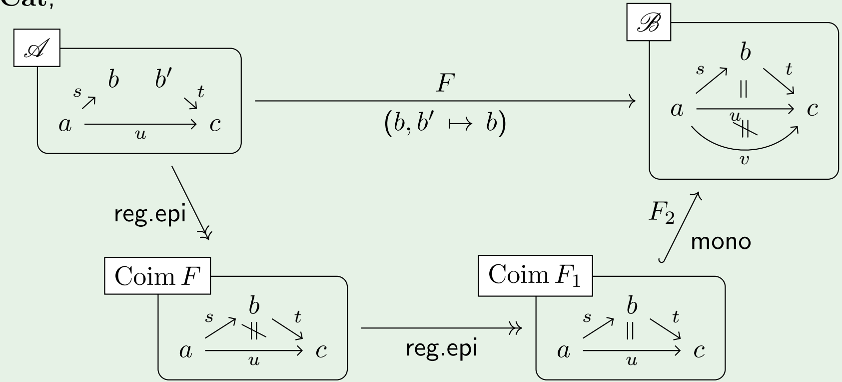
Theorem [GU71]

$\mathscr{C}$: locally κ-presentable $\implies$ $\sigma(f)$ is defined and $\sigma(f) \leq \kappa$ holds. $(\forall f$ in $\mathscr{C})$
(Equivalently, $\sigma(\mathscr{C}) \leq \kappa + 1$)

Minimum decomposition problem

Recall (a canonical decomposition in $\mathbf{Cat}$)

In $\mathbf{Cat}$,



Question

Is there a shorter decomposition of F than the canonical one in $\mathbf{Cat}$? That is, can F be decomposed as $\cdot \xrightarrow{\text{reg.epi}} \cdot \xhookrightarrow{\text{mono}} \cdot$?

Minimum decomposition problem

Definition

The minimum decomposition number $\delta(f)$:= the shortest possible length of decompositions of f. (if at least one decomposition exists)

Question (restated)

Does $\sigma(f) = \delta(f)$ hold?

Theorem (Canonical vs minimum [KN])

In a category with kernel pairs and colimits,

1. $\sigma(f)$ is defined $\iff$ $\delta(f)$ is defined.
2. Whenever they are defined, $\sigma(f) = \delta(f)$ holds.

That is, the canonical (regular) decomposition is minimum.

1. Regular epimorphisms

2. Generalized regular epimorphisms

3. Locally orthogonal factorizations

Observation

In a category with kernel pairs,
q: regular epi. $\iff$ q: the coequalizer of the kernel pair of q.

- "coequalize the kernel pair of q" $\iff$ coequalize any pair of morphisms that coequalized by q. $\cdots$ $(\star)$ (*the kernel condition associated with* q)

$\therefore$ q: regular epi. $\iff$ q: the most universal among morphisms satisfying $(\star)$.

In a category with binary coproducts,

$$X \overset{u}{\underset{v}{\rightrightarrows}} A \text{ is coequalized by } \begin{tikzcd} A \arrow[d, "q"] \\ B \end{tikzcd} \iff \begin{tikzcd} X+X \arrow[r, "{(u,v)}"] \arrow[d, "\nabla"'] & A \arrow[d, "q"] \\ X \arrow[r, dotted, "\exists"] & B \end{tikzcd}$$

$$A \xrightarrow{f} C \text{ satisfies } (\star) \iff \forall X, \begin{tikzcd} X+X \arrow[r, "\forall"] \arrow[d, "\nabla"'] & A \arrow[d, "q"'] \arrow[ddr, bend left, "f"] & \\ X \arrow[r, "\forall"] \arrow[drr, dotted, bend right, "\exists"] & B & \\ & & C \end{tikzcd}$$

Idea

Replace $\{X + X \xrightarrow{\nabla} X\}_{X \colon \mathsf{obj}}$ with another class of epimorphisms.

$\mathbf{\Lambda}$-regular epimorphisms

$\mathbf{\Lambda}$: a class of epimorphisms. Fix $A \xrightarrow{q} B$.

Definition [KN]

$A \xrightarrow{f} C$ satisfies $(\star)$
(the $\mathbf{\Lambda}$-*kernel condition assoc. with* q)
$\overset{\text{def}}{\Leftrightarrow}$

$$\begin{tikzcd}
\cdot \arrow[r, "\forall"] \arrow[d, "\mathbf{\Lambda} \ni \forall e"'] & A \arrow[d, "q"'] \arrow[dr, bend left, "f"] & \\
\cdot \arrow[r, "\forall"] \arrow[drr, dotted, bend right, "\exists"] & B & \\
 & & C
\end{tikzcd}$$

Definition [KN]

q: $\mathbf{\Lambda}$-regular epi $\overset{\text{def}}{\Leftrightarrow}$

$$\begin{tikzcd}
A \arrow[d, "q"'] \arrow[dr, bend left, "\forall f \text{ satisfying } (\star)"] & \\
B \arrow[r, dotted, "\exists!"] & \cdot
\end{tikzcd}$$

Example I

Example

$\mathbf{Pos}$: the category of posets and order-preserving maps.

$$\mathbf{\Lambda} \coloneqq \left\{ \boxed{\bullet \quad \bigstar} \xrightarrow{\iota} \boxed{\bullet < \bigstar} \quad \text{in } \mathbf{Pos} \right\}$$

$\rightsquigarrow$ $\mathbf{\Lambda}$-regular epis $=$ *surjective* order-preserving maps ($=$ epis)

Example

$\mathbf{Cat}$: the category of small categories and functors.

$$\mathbf{\Lambda} \coloneqq \left\{ \boxed{\bullet \to \bigstar} \xrightarrow{L} \boxed{\bullet \cong \bigstar} \quad \text{in } \mathbf{Cat} \right\}$$

$\rightsquigarrow$ $\mathbf{\Lambda}$-regular epis $=$ *strict localizations*

Example II

Example

$\mathbf{Met}$: the category of metric spaces and non-expansive maps.

$$\mathbf{\Lambda} := \left\{ \boxed{\bullet \overset{r+1}{\text{—}} \bigstar} \xrightarrow{e_r} \boxed{\bullet \overset{r}{\text{—}} \bigstar} \quad \text{in } \mathbf{Met} \right\}_{r \in \mathbb{Q}_{>0}}$$

$\rightsquigarrow q$: $\mathbf{\Lambda}$-regular epi $\iff d(x,y) - d(q(x), q(y)) \leq 1$ $(\forall x, y)$.

(*a shrinking at most 1*)

Example

$\mathbf{Top}$: the category of topological spaces and continuous maps.

$$\mathbf{\Lambda} := \left\{ K \xrightarrow{!_K} 1 \quad \text{in } \mathbf{Top} \right\}_{K: \text{connected space}}$$

$\rightsquigarrow q$: $\mathbf{\Lambda}$-regular epi $\iff q^{-1}(x)$: connected $(\forall x)$. (= *monotone map*)

category	$\mathbf{\Lambda}$	$\mathbf{E} = \{\mathbf{\Lambda}$-regular epis$\}$	$\mathbf{r}(\mathbf{E})$
any category with bin.coprod.	{codiagonals}	{regular epis}	{monos}
$\mathbf{Pos}$	$\{\iota\}$	{surjections}	{embeddings}
$\mathbf{Cat}$	$\{L\}$	{strict localizations}	{conservatives}
$\mathbf{Met}$	$\{e_r\}_r$	{shrinkings at most 1}	{isometries}
$\mathbf{Top}$	$\{!_K\}_K$	{monotone maps}	{*light maps*}

$\mathbf{Pos}$ admits orthogonal {surj}-factorizations.
However, neither $\mathbf{Cat}$, $\mathbf{Met}$, nor $\mathbf{Top}$ admits orthogonal $\mathbf{E}$-factorizations.

Coimages in terms of $\mathbf{\Lambda}$-regular epis

Definition

Let $A \xrightarrow{f} X$.

$$\mathcal{S} := \left\{ (e_i, u_i) \;\middle|\; \begin{tikzcd} \Gamma_i \arrow[r, "u_i"] \arrow[d, "\mathbf{\Lambda} \ni e_i"'] & A \arrow[r, "f"] & X \\ \Delta_i \arrow[urr, dotted, "\exists"'] & & \end{tikzcd} \right\}$$

Then, $\operatorname{Coim}_{\mathbf{\Lambda}} f := \operatorname{Colim} \mathcal{S}$. (multiple pushout)

$$\begin{tikzcd}
\Gamma_{i'} \arrow[rr, "u_{i'}"] \arrow[dd, "e_{i'}"'] & & A \arrow[r, "f"] \arrow[dd, "q"] & X \\
& \Gamma_i \arrow[ur, "u_i"'] \arrow[dd, "e_i"] & & \\
\Delta_{i'} \arrow[rr, "\iota_{i'}"] & & \operatorname{Coim}_{\mathbf{\Lambda}} f \arrow[uur, dotted, "\exists! f_1"'] & \\
& \Delta_i \arrow[ur, "\iota_i"'] & &
\end{tikzcd}$$

q: $\mathbf{\Lambda}$-regular epi?

Proposition [KN]

1. $\mathbf{\Lambda} \subseteq \mathbf{Reg}(\mathbf{\Lambda})$ ($:= \{\mathbf{\Lambda}$-regular epis$\}$).
2. Moreover, $\mathbf{Reg}(\mathbf{\Lambda})$ is the "multiple pushout closure" of $\mathbf{\Lambda}$. That is, q: $\mathbf{\Lambda}$-reg.epi $\iff$ q: obtained by a mult.pushout of some family of spans (e_i, u_i) s.t. $e_i \in \mathbf{\Lambda}$.

In particular,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & X \\ \mathrm{Coim}_{\mathbf{\Lambda}} f \arrow[ur, "f_1"'] & \end{tikzcd}$$

$\mathbf{\Lambda}$-regular epi $\rightsquigarrow$ q

However, $f_1 \notin \mathbf{r}(\mathbf{E})$ in general.

Replacing $\begin{pmatrix} \{\text{regular epis}\} & \rightsquigarrow & \mathbf{E} = \mathbf{Reg}(\mathbf{\Lambda}) \\ \{\text{monos}\} & \rightsquigarrow & \mathbf{r}(\mathbf{E}) \end{pmatrix}$,

Definition

$\sigma_{\mathbf{E}}(f) := \min\{\alpha \mid f_\alpha \in \mathbf{r}(\mathbf{E})\}$. (the *canonical* $\mathbf{E}$*-decomposition number*)

Example

In $\mathbf{Cat}$ with $\mathbf{E} :=$ {strict localizations},

$$\begin{tikzcd} \mathscr{A} \arrow[rr, "F"', "{(h_0\mapsto \mathsf{id}_0;\ n,n'\mapsto n)}"] \arrow[d, "\text{loc.}"] & & \mathscr{X} \\ \mathrm{Coim}_{\mathbf{\Lambda}} F \arrow[urr, "F_1"] \arrow[r, "\text{loc.}"] & \mathrm{Coim}_{\mathbf{\Lambda}} F_1 \arrow[ur, "F_2"] & \cdots \end{tikzcd}$$

$\mathscr{A}$:

$$\begin{array}{ccccccc} 0 & \xrightarrow{r_1} & 1 & \xleftarrow{s_2} & 2 & \xrightarrow{r_3} & \cdots \\ {\scriptstyle h_0}\downarrow & & & & & & \\ 0' & \underset{s_1}{\leftarrow} & 1' & \underset{r_2}{\rightarrow} & 2' & \underset{s_3}{\leftarrow} & \cdots \end{array}$$

$\mathscr{X}$:

$$0 \underset{s_1}{\overset{r_1}{\rightleftarrows}} 1 \underset{r_2}{\overset{s_2}{\leftrightarrows}} 2 \underset{s_3}{\overset{r_3}{\rightleftarrows}} \cdots \qquad (r_n\circ s_n = \mathsf{id})$$

$\mathrm{Coim}_{\mathbf{\Lambda}} F$:

$$\begin{array}{ccccccc} 0 & \xrightarrow{r_1} & 1 & \xleftarrow{s_2} & 2 & \xrightarrow{r_3} & \cdots \\ \wr\Vert & & \uparrow & & & & \\ 0' & \underset{s_1}{\leftarrow} & 1' & \underset{r_2}{\rightarrow} & 2' & \underset{s_3}{\leftarrow} & \cdots \end{array}$$

$\mathrm{Coim}_{\mathbf{\Lambda}} F_1$:

$$\begin{array}{ccccccc} 0 & \xrightarrow{r_1} & 1 & \xleftarrow{s_2} & 2 & \xrightarrow{r_3} & \cdots \\ \wr\Vert & & \wr\Vert & & \downarrow & & \\ 0' & \underset{s_1}{\leftarrow} & 1' & \underset{r_2}{\rightarrow} & 2' & \underset{s_3}{\leftarrow} & \cdots \end{array}$$

$\cdots$

$\rightsquigarrow \sigma_{\mathbf{E}}(F) = \omega.$

Theorem (Small object argument)

$\mathscr{C}$: locally κ-presentable; $\mathbf{\Lambda}$: small; $\operatorname{dom} f, \operatorname{cod} f$: κ-presentable $(\forall f \in \mathbf{\Lambda})$
$\implies \sigma_{\mathbf{E}}(\mathscr{C}) \leq \kappa + 1$, where $\mathbf{E} \coloneqq \mathbf{Reg}(\mathbf{\Lambda})$.

Corollary

$\sigma_{\mathbf{E}}(\mathbf{Cat}) = \omega + 1$, where $\mathbf{E} \coloneqq \{$strict localizations$\}$.

Minimum decomposition problem

Let $\mathbf{E} \coloneqq \mathbf{Reg}(\mathbf{\Lambda})$.

We can also generalize $\begin{pmatrix} \text{regular decomposition} & \rightsquigarrow & \mathbf{E}\text{-decomposition} \\ \text{minimum dec.num. } \delta(f) & \rightsquigarrow & \delta_{\mathbf{E}}(f) \end{pmatrix}$.

Then, $\sigma_{\mathbf{E}}$ and $\delta_{\mathbf{E}}$ still coincide:

Theorem (Canonical vs minimum [KN])

Let $\mathscr{C}$: locally small and cocomplete, $\mathbf{E} \coloneqq \mathbf{Reg}(\mathbf{\Lambda})$ with $\mathbf{\Lambda}$: small.

1. $\sigma_{\mathbf{E}}(f)$ is defined $\iff$ $\delta_{\mathbf{E}}(f)$ is defined.
2. Whenever they are defined, $\sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$ holds.

In the previous part, we did:

1. Define the class of morphisms $\mathbf{E} := \mathbf{Reg}(\boldsymbol{\Lambda})$;
2. Define the *coimage-factorizations* in terms of $\mathbf{E}$;
3. Define $\sigma_{\mathbf{E}}(f)$ as the length of the iterated coimage-factorization of f;
4. Compare $\sigma_{\mathbf{E}}(f)$ with the minimum decomposition number $\delta_{\mathbf{E}}(f)$.

Idea

Abstracting a class $\mathbf{E}$ with coimage-factorizations, we can follow the same story from 2. to 4.

Definition [MT82]

$\begin{array}{c} X \\ f\downarrow \\ Y \end{array}$ and $\begin{array}{ccc} A' & \xrightarrow{a} & A \\ & & \downarrow g \\ & & B \end{array}$ are locally orthogonal (written $f \perp^a g$) $\overset{\text{def}}{\Leftrightarrow}$

$$\begin{tikzcd} X \arrow[r, "\forall"] \arrow[d, "f"'] & A' \arrow[r, "a"] & A \arrow[d, "g"] \\ Y \arrow[urr, dotted, "\exists!"] \arrow[rr, "\forall"'] & & B \end{tikzcd}$$

RecallDefinition [MT82]

$\mathbf{E}$: iso-closed.

$\mathscr{C}$ admits locally orthogonal $\mathbf{E}$-factorizations $\overset{\text{def}}{\Leftrightarrow}$ $\forall f$ can be decomposed as

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "\mathbf{E}\ni e"'] & & \cdot \\ & \cdot \arrow[ur, "m"'] & \end{tikzcd}$$ with $\mathbf{E} \perp^e m$.

Example

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "e"'] & & \cdot \\ & \cdot \arrow[ur, "m"'] & \end{tikzcd}$$ is the coimage-factorization in terms of $\mathbf{E} = \mathbf{Reg}(\mathbf{\Lambda})$

$\iff$ it is a locally orthogonal $\mathbf{E}$-factorization, i.e., $e \in \mathbf{E}$ and $\mathbf{E} \perp^e m$.

Locally orthogonal factorizations subsume the coimage-factorization!

Theorem [Tho83; KN]

$\mathscr{C}$: co-well-powered, having small colimits and products.
TFAE for a class $\mathbf{E} \subseteq \{\text{epis}\}$:

1. $\mathscr{C}$ admits locally orthogonal $\mathbf{E}$-factorizations.
2. $\mathbf{E}$ is closed under multiple pushout.
3. $\mathbf{E} = \mathbf{Reg}(\mathbf{\Lambda})$ for some $\mathbf{\Lambda} \subseteq \{\text{epis}\}$.

Question

Is there a non-epimorphic example?

The Eilengerg–Moore decomposition (monadic towers)

Example

$\mathbf{AdjCAT}$: the category of (large) categories and left adjoints.
$\mathbf{E} := \{$monadic adjunctions$\}$.
$\rightsquigarrow$ $\mathbf{AdjCAT}$ admits locally orthogonal $\mathbf{E}$-factorizations [MT82].

$$\begin{tikzcd}
\mathscr{A} \arrow[rrr, shift left, "F"] \arrow[rrr, phantom, "\perp"] \arrow[d, shift left] \arrow[d, phantom, "\vdash"] & & & \mathscr{B} \arrow[lll, shift left, "G"] \arrow[dlll, shift left, "G_1"] \arrow[dll, shift left, "G_2"] \\
\mathscr{A}^{GF} \arrow[u, shift left] \arrow[urrr, shift left, "F_1", color=magenta] \arrow[urrr, phantom, "\perp"] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (\mathscr{A}^{GF})^{G_1F_1} \arrow[l, shift left] \arrow[urr, shift left, "F_2"] \arrow[urr, phantom, "\perp"] & & \cdots
\end{tikzcd}$$

This stabilizes at $\alpha \overset{\text{def}}{\Leftrightarrow} \mathbf{E} \perp F_\alpha \;\Leftrightarrow\; F_\alpha$: fully faithful
$\rightsquigarrow$ $\sigma_{\mathbf{E}}(F) =$ the *monadic length of* $F \dashv G$.

Minimum decomposition problem

Theorem (Canonical vs minimum, the most general [KN])

$\mathscr{C}$: a category with locally orthogonal $\mathbf{E}$-factorizations and colimits of chains.
f: a morphism in $\mathscr{C}$.

1. $\sigma_{\mathbf{E}}(f)$ is defined $\iff$ $\delta_{\mathbf{E}}(f)$ is defined.
2. $\delta_{\mathbf{E}}(f)$: 0 or successor $\implies$ $\sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$.
3. $\delta_{\mathbf{E}}(f)$: limit $\implies$ $\sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$ or $\delta_{\mathbf{E}}(f) + 1$.

Corollary

The monadic length coincides with the shortest possible length of monadic decompositions if they are finite.

Thank you!

slides $\rightsquigarrow$

This talk is based on the first half of [KN].

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Example

$\mathscr{C}$: the (l.f.p.) category of sets M with countably many partial constants $b, c_0, c_1, c_2, \ldots$ such that:

$$\Big(c_n = b \longmapsto c_{n+1}{\downarrow}\Big)_{n\geq 0}$$

$\rightsquigarrow \sigma(\mathscr{C}) = \omega + 1$

Example

$\mathscr{C}$: the (l.f.p.) category of sets M with countably many partial constants $b, c_0, c_1, c_2, \ldots$ such that:

$$\top \longmapsto b{\downarrow}$$

$$\Big(c_n{\downarrow} \longmapsto c_{n+1}{\downarrow}\Big)_{n\geq 0}$$

$$\Big(c_{n+1} = b \longmapsto c_n{\downarrow}\Big)_{n\geq 0}$$

$\rightsquigarrow \sigma(\mathscr{C}) = \omega$