Universal algebra over locally presentable categories

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← Today's slides

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Relativization via monads

Theorem ([Lin69])

There is an equivalence

$$\mathbf{Th}^S \simeq \mathbf{Mnd}_{\mathrm{f}}(\mathbf{Set}^S).$$

Here,

$\mathbf{Th}^S$: the category of S-sorted equational theories,

$\mathbf{Mnd}_{\mathrm{f}}(\mathbf{Set}^S)$: the category of finitary monads on $\mathbf{Set}^S$.

S-sorted equational theory $=$ finitary monad on $\mathbf{Set}^S$

$\downarrow$ generalize

$\mathscr{A}$-relative algebraic theory $=$ κ-ary monad on $\mathscr{A}$

($\mathscr{A}$: a locally κ-presentable category)

Relative algebraic theories

Informal definition [Kaw23]

$\mathscr{A}$: a (locally presentable) category

An $\mathscr{A}$-relative algebraic theory consists of:

- a set Ω of partial operators;
- a set E of implications $\cdots\left(\underbrace{\underline{\text{YYY}}}_{\text{postcondition}} \text{ whenever } \underbrace{\underline{\text{XXX}}}_{\text{precondition}}\right)$

such that

- For each operator $\omega \in \Omega$, its domain must be defined by "$\mathscr{A}$'s language."
- For each implication in E, its precondition must be written in "$\mathscr{A}$'s language."

$\mathbf{Set}^S$-relative algebraic theories $=$ S-sorted equational theories

A generalized Linton theorem

Theorem ([Kaw23; Kaw24])

For a locally κ-presentable category $\mathscr{A}$, there is an equivalence

$$\mathbf{Th}_{\kappa}^{\mathscr{A}} \simeq \mathbf{Mnd}_{\kappa}(\mathscr{A}).$$

Here,

$\mathbf{Th}_{\kappa}^{\mathscr{A}}$: the category of $\mathscr{A}$-relative (κ-ary) algebraic theories,

$\mathbf{Mnd}_{\kappa}(\mathscr{A})$: the category of κ-ary monads on $\mathscr{A}$.

Example: small categories

Example

A small category consists of:

- a base quiver $\operatorname{mor}\mathscr{C} \underset{\mathrm{c}}{\overset{\mathrm{d}}{\rightrightarrows}} \operatorname{ob}\mathscr{C}$;
- a total operator $\mathrm{id}\colon \operatorname{ob}\mathscr{C} \to \operatorname{mor}\mathscr{C}$;
- a partial operator $\circ\colon \operatorname{mor}\mathscr{C} \times \operatorname{mor}\mathscr{C} \to \operatorname{mor}\mathscr{C}$ such that

$$g \circ f \text{ is defined iff } \mathrm{d}(g) = \mathrm{c}(f)$$

which satisfy the following:

- $\mathrm{d}(\mathrm{id}(x)) = x$ and $\mathrm{c}(\mathrm{id}(x)) = x$;
- $\mathrm{d}(g \circ f) = \mathrm{d}(f)$ and $\mathrm{c}(g \circ f) = \mathrm{c}(g)$ whenever $\mathrm{d}(g) = \mathrm{c}(f)$;
- $f \circ \mathrm{id}(\mathrm{d}(f)) = f$ and $\mathrm{id}(\mathrm{c}(f)) \circ f = f$;
- $(h \circ g) \circ f = h \circ (g \circ f)$ whenever $\mathrm{d}(h) = \mathrm{c}(g)$ and $\mathrm{d}(g) = \mathrm{c}(f)$.

Small categories are algebras over quivers.

Further examples

Example

		algebras over $\rightsquigarrow$
small categories	$\rightsquigarrow$	quivers
UDO semirings	$\rightsquigarrow$	posets
partial Boolean algebras	$\rightsquigarrow$	graphs
monoid-graded rings	$\rightsquigarrow$	monoid-graded sets
generalized complete metric spaces	$\rightsquigarrow$	generalized metric spaces
Banach spaces	$\rightsquigarrow$	pointed metric spaces

A technical remark

Definition ([PV07])

- (κ-ary) partial Horn theory $\cdots$ a logical theory based on multi-sorts, partial functions, relations, and (partial) Horn implications.
- $\mathbf{Mod}\,\mathbb{S}$ $\cdots$ the category of models of a partial Horn theory $\mathbb{S}$.

Theorem ([PV07])

TFAE for a category $\mathscr{A}$:

1. $\mathscr{A}$ is locally κ-presentable.
2. $\mathscr{A} \simeq \mathbf{Mod}\,\mathbb{S}$ for some κ-ary partial Horn theory $\mathbb{S}$.

We actually define $\mathbb{S}$-relative algebraic theories for partial Horn theories $\mathbb{S}$.

$$\rightsquigarrow \quad \mathscr{A}\text{-rel. alg. theory} \;=\; \mathbb{S}\text{-rel. alg. theory where } \mathscr{A} \simeq \mathbf{Mod}\,\mathbb{S}.$$

Birkhoff's variety theorem

Birkhoff's variety theorem [Bir35]

(Ω, E): a single-sorted algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable by equations.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under products, subobjects, and quotients.

closed under products: $A_i \in \mathscr{E} \implies \prod_i A_i \in \mathscr{E}$.

closed under subobjects: $B \subseteq A$: sub, $A \in \mathscr{E} \implies B \in \mathscr{E}$.

closed under quotients: $A \twoheadrightarrow B$: surj, $A \in \mathscr{E} \implies B \in \mathscr{E}$.

A generalized Birkhoff's theorem

Theorem ([Kaw23; Kaw24])

(Ω, E): an $\mathscr{A}$-relative (κ-ary) algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under products, closed subobjects, (U, κ)-pure quotients, and κ-filtered colimits.

single-sorted alg. ($\mathbf{Set}$-relative alg.)		$\mathscr{A}$-relative alg.
products	$\rightsquigarrow$	products
subobjects	$\rightsquigarrow$	*closed subobjects*
quotients	$\rightsquigarrow$	(U, κ)*-pure quotients*
	$\rightsquigarrow$	κ*-filtered colimits* (new)

What are closed subobjects and (U, κ)-pure quotients?

$$\mathscr{A} \quad \cdots \quad \text{a locally } \kappa\text{-presentable category}$$
$$(\Omega, E) \quad \cdots \quad \text{an } \mathscr{A}\text{-rel. alg. theory}$$
$$\mathbf{Alg}(\Omega, E) \xrightarrow{U} \mathscr{A} \quad \cdots \quad \text{the forgetful functor}$$

Informal definition

1. A subalg. $B \subseteq A$ in $\mathbf{Alg}(\Omega, E)$ is closed if:
   - For every relation R in "the language of $\mathscr{A}$,"

$$R(\vec{b}) \text{ holds in } UA \quad \Rightarrow \quad R(\vec{b}) \text{ holds in } UB.$$

2. $A \xrightarrow{p} B$ in $\mathbf{Alg}(\Omega, E)$ is a (U, κ)-pure quotient if:
   - For every κ-ary formula in "the language of $\mathscr{A}$,"

$$\varphi(\vec{b}) \text{ holds in } UB \quad \Rightarrow \quad \exists \vec{a} \xmapsto{Up} \vec{b} \text{ s.t. } \varphi(\vec{a}) \text{ holds in } UA.$$

Example

$$\begin{array}{rcl} \mathbf{Pos} & \cdots & \text{the category of posets.} \\ (\Omega, \varnothing) & \cdots & \text{a } \mathbf{Pos}\text{-rel. alg. theory defined by} \\ & & \Omega := \{\ominus\}, \quad x \ominus y \text{ is defined iff } x \geq y. \\ \mathbf{Alg}(\Omega, \varnothing) \xrightarrow{U} \mathbf{Pos} & \cdots & \text{the forgetful functor.} \end{array}$$

In $\mathbf{Alg}(\Omega, \varnothing)$, under $x \ominus y := x - y$ in $\mathbb{N}$,

- $\{0 < 2 \quad 3\} \subseteq \{0 < 1 < 2 < \cdots\}$ $\cdots$ subalgebla, but not closed.
- $\{0 < 2 < 4\} \subseteq \{0 < 1 < 2 < \cdots\}$ $\cdots$ closed subalgebla.
- $\left\{ \begin{array}{cc} 0 & 0 \\ \wedge & \\ 1 & 1 \\ & \wedge \\ & 2 \end{array} \right\} \to \left\{ \begin{array}{c} 0 \\ \wedge \\ 1 \\ \wedge \\ 2 \end{array} \right\}$ $\cdots$ surjection, but not a $(U, \aleph_0)$-pure quotient.
- $\left\{ \begin{array}{cccc} 0 & 0 & 0 & \cdots \\ & \wedge & \wedge & \\ & 1 & 1 & \cdots \\ & & \wedge & \\ & & 2 & \cdots \\ & & & \\ & & & \ddots \end{array} \right\} \to \left\{ \begin{array}{c} 0 \\ \wedge \\ 1 \\ \wedge \\ 2 \\ \wedge \\ \vdots \end{array} \right\}$ $\cdots$ $(U, \aleph_0)$-pure quotient, but not $(U, \aleph_1)$-pure quotient.

Filtered colimits are necessary

Example ($\mathbf{Set}^{\mathbb{N}}$-relative algebra [ARV12])

$$\mathscr{E} := \{1\} \cup \{A \in \mathbf{Set}^{\mathbb{N}} \mid \exists m \in \mathbb{N}.\ A_m = \varnothing\}.$$

$\mathscr{E} \subseteq \mathbf{Set}^{\mathbb{N}}$ is closed under...

- ✓ products
- ✓ closed subobjects = sort-wise injections
- ✓ pure quotients = sort-wise surjections
- × filtered colimits

$$\begin{array}{ccccccccc}
A_0 & := & (\varnothing, & \varnothing, & \varnothing, & \varnothing, & \cdots &) & \in \mathscr{E} \\
\cap & & \cap & \cap & \cap & \cap & & & \\
A_1 & := & (2, & \varnothing, & \varnothing, & \varnothing, & \cdots &) & \in \mathscr{E} \\
\cap & & \cap & \cap & \cap & \cap & & & \\
A_2 & := & (2, & 2, & \varnothing, & \varnothing, & \cdots &) & \in \mathscr{E} \\
\cap & & \cap & \cap & \cap & \cap & & & \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & & \\
\operatorname*{Colim}_{n \in \omega} A_n & = & (2, & 2, & 2, & 2, & \cdots &) & \notin \mathscr{E}
\end{array}$$

Example ($\mathbf{Set}_\omega$-relative algebra [Kaw25])

$\mathbf{Set}_\omega$ $\cdots$ the category of sets with countably many constants $(c_n)_n$.

$$\mathscr{E} := \{1\} \cup \{A \in \mathbf{Set}_\omega \mid \exists i, j \text{ s.t. } c_i \neq c_j \text{ in } A\}.$$

$\mathscr{E} \subseteq \mathbf{Set}_\omega$ is closed under...

- ✓ products
- ✓ closed subobjects = subalgebras
- ✓ pure quotients = surjections that do not merge any constants
- × filtered colimits

$$\begin{array}{cccccccccc}
A_0 & := & \{c_0, & c_1, & c_2, & c_3, & \cdots, & \infty\} & \in \mathscr{E} \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
A_1 & := & \{c_0 = & c_1, & c_2, & c_3, & \cdots, & \infty\} & \in \mathscr{E} \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
A_2 & := & \{c_0 = & c_1 = & c_2, & c_3, & \cdots, & \infty\} & \in \mathscr{E} \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \\
\operatorname*{Colim}_{n \in \omega} A_n & := & \{c_0 = & c_1 = & c_2 = & c_3 = & \cdots, & \infty\} & \notin \mathscr{E}
\end{array}$$

The filtered colimit elimination problem

However, filtered colimits are not required for $\mathbf{Set}$-rel. alg. in Birkhoff's theorem.

Question

Why can filtered colimits be eliminated in the case of $\mathbf{Set}$-relative algebras?

Answer

The category $\mathbf{Set}$ satisfies a "noetherian" condition.

A noetherian condition for categories

Definition ([Kaw25])

A category $\mathscr{A}$ satisfies the ascending chain condition (ACC) if it has no chain $A_0 \to A_1 \to A_2 \to \cdots$ of objects such that there is no morphism $A_n \leftarrow A_{n+1}$ for all n.

Example

$\mathbf{Set}$ satisfies ACC.

Proof.

Let $A_0 \to A_1 \to \cdots$ be an ω-chain of sets.If there is no map $A_0 \leftarrow A_1$, then $A_0 = \varnothing$ and $A_1 \neq \varnothing$.Thus, a map $A_1 \leftarrow A_2$ exists. □

More generally,

Proposition

$$\mathbf{Set}^S \text{ satisfies ACC } \Leftrightarrow \text{ the set } S \text{ is finite.}$$

Filtered colimit elimination

Theorem ([Kaw25; Kaw24])

(Ω, E): an $\mathscr{A}$-relative (κ-ary) algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
If $\mathscr{A}$ satisfies ACC,
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under products, closed subobjects, (U, κ)-pure quotients, ~~and κ-filtered colimits~~.

Some applications of filtered colimit elimination

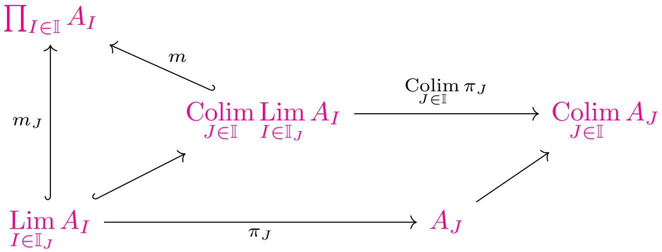
Corollary

- $\mathbf{Set}$ satisfies ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for single-sorted alg.
  - $\rightsquigarrow$ The classical Birkhoff theorem [Bir35]
- $\mathbf{Set}^n$ satisfied ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for finite-sorted alg.
  - $\rightsquigarrow$ This subsumes a result in [ARV12].
- $\mathbf{Pos}$ satisfied ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for ordered alg.
  - $\rightsquigarrow$ This subsumes a result in [Blo76].
- $\mathbf{Met}_\infty$, the category of generalized metric spaces, satisfied ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for metric alg.
  - $\rightsquigarrow$ This subsumes a result in [Hin16].

Filtered colimit elimination: sketch of proof

fullsub $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: closed under products, closed sub, (U, κ)-pure quo.
$(A_J)_{J \in \mathbb{I}}$: a κ-filtered diagram s.t. $A_J \in \mathscr{E}$.

For each $J \in \mathbb{I}$, we can construct a "nice" wide sub-diagram $J \in \mathbb{I}_J \subseteq \mathbb{I}$.



in $\mathbf{Alg}(\Omega, E)$

$\rightsquigarrow$ $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under κ-filtered colimits. □

Weak converse

Theorem ([Kaw25])

$\mathscr{A}$: a l.f.p. category. Assume that, for every fullsub. of $\mathscr{A}$, closure under filtered colimits follows from the others: $\mathbf{P}$(products), $\mathbf{S}$(closed sub), $\mathbf{H}$($\aleph_0$-pure quo). Then,

1. The fullsub $\mathscr{A}_{\mathrm{fp,c}} := \{$finitely presentable connected objs$\} \subseteq \mathscr{A}$ satisfies ACC.
2. If $\varnothing \xrightarrow{!} 1$ in $\mathscr{A}$ is strongly monic, the fullsub $\mathscr{A}_{\mathrm{fp}} := \{$finitely presentable objs$\} \subseteq \mathscr{A}$ satisfies ACC.

Sketch of proof:
Let $A_0 \to A_1 \to A_2 \to \cdots$ in $\mathscr{A}_{\mathrm{fp}}$. Consider

$$\mathscr{E} := \{X \mid \exists n.\ X \xrightarrow{\exists} A_n\} \quad \subseteq \mathscr{A}.$$

Using finite presentability, its $\mathbf{HSP}$-closure can be computed as

$$\mathbf{HSP}(\mathscr{E}) = \mathbf{S}(1) \cup \mathbf{H}(\mathscr{E}).$$

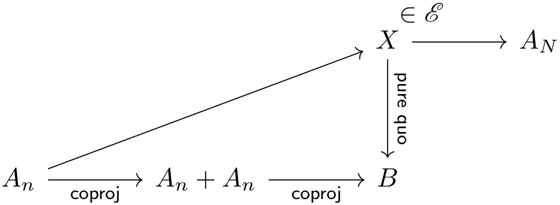
Since $A_n + A_n \in \mathscr{E}$ $(\forall n)$, $B := \operatorname*{Colim}_{n\in\omega}(A_n + A_n) \in \mathbf{HSP}(\mathscr{E})$.
The additional conditions ensure that $B \notin \mathbf{S}(1)$. $\quad \therefore B \in \mathbf{H}(\mathscr{E})$

$$A_0 \to A_1 \to A_2 \to \cdots \quad \text{in } \mathscr{A}_{\text{fp}}$$

$$\mathscr{E} := \{X \mid \exists n.\ X \overset{\exists}{\to} A_n\} \quad \subseteq \mathscr{A}$$

$$B := \operatorname*{Colim}_{n \in \omega}(A_n + A_n) \ \in \mathbf{H}(\mathscr{E})$$

Thus, we have:



$$(\forall n \geq N)$$

Thus, $A_0 \to A_1 \to A_2 \to \cdots$ eventually "stabilizes." □

Open problems

Open problem 1

Is there any locally presentable category that satisfies filtered colimit elimination but not ACC?

More precisely, is there any κ-ary partial Horn theory $\mathbb{S}$ for some κ that satisfies the following conditions?

- Every full subcategory of $\mathbf{Mod}\,\mathbb{S}$ is closed under κ-filtered colimits whenever it is closed under products, $\mathbb{S}$-closed subobjects, and κ-pure quotients.
- The category $\mathbf{Mod}\,\mathbb{S}$ does not satisfy ACC.

The next one is weaker than 1 and independent of partial Horn theories:

Open problem 2

Is there any locally finitely presentable category that does not satisfy ACC but satisfies it for the full subcategory of finitely presentable objects?

Thank you!

Today's slides

My homepage

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