Universal algebra over locally presentable categories

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Single-sorted algebras

Definition

A (single-sorted) algebra consists of:

- a base set A;
- operators $\sigma\colon A^n \to A$ $(n \geq 0)$;
- equations.

Example

A group consists of:

- a base set G;
- operators $e\colon 1 \to G$, $\quad i\colon G \to G$, $\quad m\colon G^2 \to G$;
- equations $m(e,x) = x = m(x,e)$, $\quad m(x,i(x)) = e = m(i(x),x)$, $m(m(x,y),z) = m(x,m(y,z))$.

Multi-sorted algebras

Definition

S: a set. (the set of sorts)
An S-sorted algebra consists of:

- base sets $(A_s)_{s \in S}$ indexed by S;
- operators $\sigma \colon A_{s_1} \times \cdots \times A_{s_n} \to A_s$;
- equations.

Example

A chain complex consists of:

- base sets $(A_n)_{n \in \mathbb{Z}}$;
- operators $0_n \colon 1 \to A_n, \quad -_n \colon A_n \to A_n, \quad +_n \colon A_n \times A_n \to A_n,$
  $d_n \colon A_n \to A_{n+1}$;
- appropriate equations.

This is an $\mathbb{Z}$-sorted algebra.

The free-forgetful adjunctions

$$\begin{array}{c}\mathbf{Grp}\\ F\uparrow \dashv \downarrow U\\ \mathbf{Set}\end{array} \qquad\qquad \begin{array}{c}\mathbf{Ch}\\ F\uparrow \dashv \downarrow U\\ \mathbf{Set}^{\mathbb{Z}}\end{array}$$

$$\begin{array}{c}\mathbf{Alg}(\Omega, E)\\ F\uparrow \dashv \downarrow U\\ \mathbf{Set}^{S}\end{array}$$

$(\underbrace{\Omega}_{\text{operators}}, \underbrace{E}_{\text{equations}})$: an S-sorted algebraic theory.

Relativization via monads

Theorem ([Lin69])

There is an equivalence

$$\mathbf{Th}^S \simeq \mathbf{Mnd}_{\mathrm{f}}(\mathbf{Set}^S).$$

Here,

$\mathbf{Th}^S$: the category of S-sorted algebraic theories,

$\mathbf{Mnd}_{\mathrm{f}}(\mathbf{Set}^S)$: the category of finitary monads on $\mathbf{Set}^S$.

S-sorted algebraic theory $=$ finitary monad on $\mathbf{Set}^S$

$\downarrow$ generalize

??? $\mathscr{A}$-relative algebraic theory $=$ κ-ary monad on $\mathscr{A}$

($\mathscr{A}$: a locally κ-presentable category)

Relative algebraic theories

Informal definition [Kaw23a]

$\mathscr{A}$: a (locally presentable) category

An $\mathscr{A}$-relative algebraic theory consists of:

- a set Ω of partial operators;
- a set E of implications $\cdots$ ($\underbrace{\text{YYY}}_{\text{postcondition}}$ whenever $\underbrace{\text{XXX}}_{\text{precondition}}$)

such that

- For each operator $\omega \in \Omega$, its domain must be defined by "$\mathscr{A}$'s language."
- For each implication in E, its precondition must be written in "$\mathscr{A}$'s language."

A generalized Linton theorem

Theorem ([Kaw23a; Kaw24])

For a locally κ-presentable category $\mathscr{A}$, there is an equivalence

$$\mathbf{Th}_{\kappa}^{\mathscr{A}} \simeq \mathbf{Mnd}_{\kappa}(\mathscr{A}).$$

Here,

$\mathbf{Th}_{\kappa}^{\mathscr{A}}$: the category of $\mathscr{A}$-relative (κ-ary) algebraic theories,

$\mathbf{Mnd}_{\kappa}(\mathscr{A})$: the category of κ-ary monads on $\mathscr{A}$.

$\uparrow$ generalize

Recall (Linton's theorem)

$$\mathbf{Th}_{\aleph_0}^{S} \simeq \mathbf{Mnd}_{\aleph_0}(\mathbf{Set}^{S}).$$

Example: small categories

Example

A small category consists of:

- a base quiver $\mathrm{mor}\mathscr{C} \overset{\mathrm{d}}{\underset{\mathrm{c}}{\rightrightarrows}} \mathrm{ob}\mathscr{C}$;
- a total operator $\mathrm{id}\colon \mathrm{ob}\mathscr{C} \to \mathrm{mor}\mathscr{C}$;
- a partial operator $\circ\colon \mathrm{mor}\mathscr{C} \times \mathrm{mor}\mathscr{C} \to \mathrm{mor}\mathscr{C}$ such that

$$g \circ f \text{ is defined iff } \mathrm{d}(g) = \mathrm{c}(f)$$

which satisfy the following:

- $\mathrm{d}(\mathrm{id}(x)) = x$ and $\mathrm{c}(\mathrm{id}(x)) = x$;
- $\mathrm{d}(g \circ f) = \mathrm{d}(f)$ and $\mathrm{c}(g \circ f) = \mathrm{c}(g)$ whenever $\mathrm{d}(g) = \mathrm{c}(f)$;
- $f \circ \mathrm{id}(\mathrm{d}(f)) = f$ and $\mathrm{id}(\mathrm{c}(f)) \circ f = f$;
- $(h \circ g) \circ f = h \circ (g \circ f)$ whenever $\mathrm{d}(h) = \mathrm{c}(g)$ and $\mathrm{d}(g) = \mathrm{c}(f)$.

Small categories are algebras over quivers.

Further examples

Example

		algebras over $\sim$
small categories	$\rightsquigarrow$	quivers
UDO semirings	$\rightsquigarrow$	posets
partial Boolean algebras	$\rightsquigarrow$	graphs
monoid-graded rings	$\rightsquigarrow$	monoid-graded sets
generalized complete metric spaces	$\rightsquigarrow$	generalized metric spaces
Banach spaces	$\rightsquigarrow$	pointed metric spaces

Equational classes

Definition

(Ω, E): a single-sorted algebraic theory. A full subcategory $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable (by equations) if $\mathscr{E} = \mathbf{Alg}(\Omega, E + {}^{\exists}E')$, i.e., $\mathscr{E}$ can be defined by adding equations.

Example

$\{$commutative monoids$\} \subseteq \mathbf{Mon}$ is definable by the equation $xy = yx$.

Example

$\{$invertible monoids$\} \subseteq \mathbf{Mon}$ is not definable by equations.not definable by equations.

How can we prove this?

Birkhoff's variety theorem

Birkhoff's variety theorem [Bir35]

(Ω, E): a single-sorted algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable by equations.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under products, subobjects, and quotients.

closed under products: $A_i \in \mathscr{E} \implies \prod_i A_i \in \mathscr{E}$.

closed under subobjects: $B \subseteq A$: sub, $A \in \mathscr{E} \implies B \in \mathscr{E}$.

closed under quotients: $A \twoheadrightarrow B$: surj, $A \in \mathscr{E} \implies B \in \mathscr{E}$.

Corollary

$\{$invertible monoids$\} \subseteq \mathbf{Mon}$ is not definable by equations.

Proof.

$\underbrace{\mathbb{N}}_{\neg\text{invertible}} \subset \underbrace{\mathbb{Z}}_{\text{invertible}} \rightsquigarrow \{$inv. monoids$\} \subseteq \mathbf{Mon}$: not closed under subobjects $\square$

A generalized Birkhoff's theorem

Theorem ([Kaw23a; Kaw24])

(Ω, E): an $\mathscr{A}$-relative (κ-ary) algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under products, closed subobjects, (U, κ)-local retracts, and κ-filtered colimits.

single-sorted alg. ($\mathbf{Set}$-relative alg.)		$\mathscr{A}$-relative alg.
products	$\rightsquigarrow$	products
subobjects	$\rightsquigarrow$	*closed subobjects*
quotients	$\rightsquigarrow$	(U, κ)-*local retracts*
	$\rightsquigarrow$	κ-*filtered colimits* (new)

The filtered colimit elimination problem

Question

Why can the closure property under filtered colimits be eliminated in the case of $\mathbf{Set}$-relative algebras?

Answer

The category $\mathbf{Set}$ satisfies a “noetherian” condition.

A noetherian condition for categories

Definition ([Kaw23b])

A category $\mathscr{A}$ satisfies the ascending chain condition (ACC) if it has no chain $A_0 \to A_1 \to A_2 \to \cdots$ of objects such that there is no morphism $A_n \leftarrow A_{n+1}$ for all n.

Example

$\mathbf{Set}$ satisfies ACC.

Proof.

Let $A_0 \to A_1 \to \cdots$ be an ω-chain of sets.If there is no map $A_0 \leftarrow A_1$, then $A_0 = \varnothing$ and $A_1 \neq \varnothing$.Thus, a map $A_1 \leftarrow A_2$ exists. □

Example

Quiv, the category of quivers, does not satisfy ACC.

Proof.

Let Q_n denote the n-path

$$Q_n\colon \qquad 0 \to 1 \to 2 \to \cdots \to n.$$

Then, the inclusions yields a chain $Q_0 \to Q_1 \to Q_2 \to \cdots$, and there is no quiver morphism $Q_n \leftarrow Q_{n+1}$. □

Example

Ring, the category of rings, does not satisfy ACC.

Proof.

This is because there is a non-trivial chain of finite fields

$$\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2} \hookrightarrow \mathbb{F}_{p^4} \hookrightarrow \cdots \hookrightarrow \mathbb{F}_{p^{2^n}} \hookrightarrow \cdots .$$

□

Relation to ordinary ACC

Definition

- Objects X and Y are strongly connected if there are morphisms $X \to Y$, $Y \to X$.
- An equivalence class under strong connectedness is called a strongly connected component.
- $\sigma(\mathscr{A})$: the large poset of all strongly connected components in a category $\mathscr{A}$. (the *posetification* of $\mathscr{A}$)

$$\mathbf{CAT} \underset{\text{inclusion}}{\overset{\sigma}{\underset{\perp}{\rightleftarrows}}} \mathbf{POS}$$

Proposition

A category $\mathscr{A}$ satisfies ACC $\Leftrightarrow$ the large poset $\sigma(\mathscr{A})$ satisfies ACC.

Proposition

$\mathbf{Set}^S$ satisfies ACC $\quad\Leftrightarrow\quad$ the set S is finite.

Proof.

Since the posetification σ preserves products, the following holds:

$$\sigma(\mathbf{Set}^S) \cong \sigma(\mathbf{Set})^S \cong \{0 < 1\}^S \cong \mathscr{P}(S).$$

"$\mathscr{P}(S)$ satisfies ACC $\Leftrightarrow$ S: finite" is trivial. □

Filtered colimit elimination

Theorem ([Kaw23b; Kaw24])

(Ω, E): an $\mathscr{A}$-relative (κ-ary) algebraic theory. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: fullsub.
Assume that $\mathscr{A}$ satisfies ACC.
TFAE:

1. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is definable.
2. $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under <u>products</u>, <u>closed subobjects</u>, <u>(U, κ)-local retracts</u>, ~~and κ-filtered colimits~~.

Some applications of filtered colimit elimination

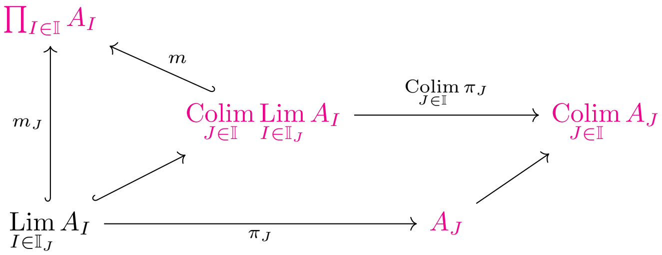
Corollary

- $\mathbf{Set}$ satisfies ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for single-sorted alg.
  - $\rightsquigarrow$ The classical Birkhoff theorem [Bir35]
- $\mathbf{Set}^n$ satisfied ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for finite-sorted alg.
  - $\rightsquigarrow$ This subsumes a result in [ARV12].
- $\mathbf{Pos}$ satisfied ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for ordered alg.
  - $\rightsquigarrow$ This subsumes a result in [Blo76].
- $\mathbf{Met}_\infty$, the category of generalized metric spaces, satisfied ACC.
  - $\rightsquigarrow$ fil.colim.elim. holds for metric alg.
  - $\rightsquigarrow$ This subsumes a result in [Hin16].

Filtered colimit elimination: sketch of proof

fullsub $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$: closed under products, closed sub, (U, κ)-local ret.
$(A_J)_{J \in \mathbb{I}}$: a κ-filtered diagram s.t. $A_J \in \mathscr{E}$.

For each $J \in \mathbb{I}$, we can construct a "nice" wide sub-diagram $\mathbb{I}_J \subseteq \mathbb{I}$.



in $\mathbf{Alg}(\Omega, E)$

$\rightsquigarrow$ $\mathscr{E} \subseteq \mathbf{Alg}(\Omega, E)$ is closed under κ-filtered colimits. □

Locally connected categories

Definition

$C \in \mathscr{C}$ is connected $\overset{\text{def}}{\Leftrightarrow}$ $\mathscr{C}(C, \bullet)\colon \mathscr{C} \to \mathbf{Set}$ preserves small coproducts.

Example

1. A top. space $X \in \mathbf{Top}$ is connected $\Leftrightarrow$ it is connected (in the usual sense).
2. A set $X \in \mathbf{Set}$ is connected $\Leftrightarrow$ it is a singleton.
3. A category $\mathscr{C} \in \mathbf{Cat}$ is connected $\Leftrightarrow$ all objects are connected by zig-zags.
4. A presheaf $P \in \mathbf{Set}^{\mathscr{C}^{\text{op}}}$ is connected $\Leftrightarrow$ so is the caty of elements $\int P$.

Definition

$\mathscr{C}$ is locally connected $\overset{\text{def}}{\Leftrightarrow}$ it has small coproducts and every object is a small coproduct of connected objects.

Example

- $\mathbf{Top}$ is not locally connected.
- $\mathbf{Set}$, $\mathbf{Cat}$, and any presheaf categories $\mathbf{Set}^{\mathscr{C}^{\text{op}}}$ are locally connected.

A characterization of locally connected categories

Definition

Given a category $\mathscr{A}$, we define a category $\mathbf{Fam}(\mathscr{A})$ (*the category of families*):

- object $\cdots$ a small family $(A_i \in \mathscr{A})_{i \in I}$;
- morphism $(A_i)_I \to (B_j)_J$ $\cdots$ a map $I \xrightarrow{f} J$ together with a family $(A_i \xrightarrow{f_i} B_{f(i)} \text{ in } \mathscr{A})_{i \in I}$.

Theorem ([CV98])

$\mathscr{C}$ is locally connected $\Leftrightarrow$ $\mathscr{C} \simeq \mathbf{Fam}(\mathscr{A})$ for some $\mathscr{A}$.

$\mathscr{C}$: locally connected $\rightsquigarrow$ $\mathscr{C} \simeq \mathbf{Fam}(\mathscr{C}_{\mathrm{conn}})$

($\mathscr{C}_{\mathrm{conn}} \subseteq \mathscr{C}$: the fullsub of all connected objects)

ACC for locally connected categories

Definition

- $L \subseteq \mathrm{ob}\mathscr{A}$ is called a lower class $\overset{\text{def}}{\Leftrightarrow}$ "$X \to Y \in L$" implies $X \in L$.
- $\mathbb{L}(\mathscr{A})$: the (large) poset of lower classes on $\mathscr{A}$.

Lemma ([Kaw23b])

$\mathscr{C}$: locally connected $+\alpha$ $\rightsquigarrow$ $\sigma(\mathscr{C}) \cong \mathbb{L}(\mathscr{C}_{\mathrm{conn}})$ $(\cong \mathbb{L}\sigma(\mathscr{C}_{\mathrm{conn}}))$.

Proof.

$$\sigma(\mathscr{C}) \cong \sigma(\mathbf{Fam}(\mathscr{C}_{\mathrm{conn}})) \cong \mathbb{L}(\mathscr{C}_{\mathrm{conn}}).$$

□

Lemma ([Kaw23b])

The poset $\mathbb{L}(\mathscr{A})$ satisfies ACC $\Leftrightarrow$ Every lower class on $\mathscr{A}$ is *finitely generated*.

Corollary ([Kaw23b])

A locally connected category $\mathscr{C}$ satisfies ACC $\Leftrightarrow$ Every lower class on $\mathscr{C}_{\mathrm{conn}}$ is finitely generated.

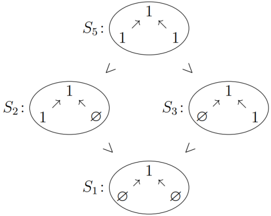
$\mathbf{Cospan} := \mathbf{Set}^{[\cdot\to\cdot\leftarrow\cdot]}$ (presheaf category)
$\mathbf{Cospan}$ has only 6 strongly connected components:

$$S_0: \begin{array}{ccc} & \varnothing & \\ \nearrow & & \nwarrow \\ \varnothing & & \varnothing, \end{array} \quad S_1: \begin{array}{ccc} & 1 & \\ \nearrow & & \nwarrow \\ \varnothing & & \varnothing, \end{array} \quad S_2: \begin{array}{ccc} & 1 & \\ \nearrow & & \nwarrow \\ 1 & & \varnothing, \end{array}$$

$$S_3: \begin{array}{ccc} & 1 & \\ \nearrow & & \nwarrow \\ \varnothing & & 1, \end{array} \quad S_4: \begin{array}{ccc} & 2 & \\ \ulcorner 0 \urcorner \nearrow & & \nwarrow \ulcorner 1 \urcorner \\ 1 & & 1, \end{array} \quad S_5: \begin{array}{ccc} & 1 & \\ \nearrow & & \nwarrow \\ 1 & & 1. \end{array}$$

On the other hand,

$\sigma(\mathbf{Cospan}_{\mathrm{conn}}) =$



$\sigma(\mathbf{Cospan}) \cong \mathbb{L}(\sigma(\mathbf{Cospan}_{\mathrm{conn}}))$ is displayed as follows:

$$\begin{array}{ccc} & S_5 & \\ & \vee & \\ & S_4 & \\ \angle & & \searrow \\ S_2 & & S_3 \\ \searrow & & \angle \\ & S_1 & \\ & \vee & \\ & S_0 & \end{array} \quad \cong \quad \begin{array}{ccc} & \downarrow S_5 & \\ & \cup & \\ & \downarrow\{S_2, S_3\} & \\ \subset & & \supset \\ \downarrow S_2 & & \downarrow S_3 \\ \supset & & \subset \\ & \downarrow S_1 & \\ & \cup & \\ & \varnothing & \end{array}$$

ACC for G-$\mathbf{Set}$

G: a topological group $\rightsquigarrow$ G-$\mathbf{Set}$: locally connected

Definition

$A \in \mathscr{E}$ is called an atom $\overset{\text{def}}{\Leftrightarrow}$ $A \neq 0$ and $\mathrm{Sub}(A) = \{0, A\}$.

$$(G\text{-}\mathbf{Set})_{\mathrm{conn}} = \{\text{atoms in } G\text{-}\mathbf{Set}\} \simeq \{\text{open subgroups of } G\}$$

Corollary ([Kaw23b])

G: a topological group

1. $\sigma(G\text{-}\mathbf{Set}) \cong \mathbb{L}(\text{open subgroups of } G)$
2. G-$\mathbf{Set}$ satisfies ACC $\Leftrightarrow$ Every lower set of open subgroups of G is finitely generated.

Thank you!

Today's slide

My homepage

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κ-filtered colimits

Definition

A small category $\mathbb{I}$ is κ-filtered if every $(< \kappa)$-small diagram has a cocone in $\mathbb{I}$.

Definition

A κ-filtered colimit is a colimit of a functor from a κ-filtered small category.

Representing models

Theorem ([Kaw23a; Kaw24])

$\mathbb{T}$: a κ-ary partial Horn theory
For every $\mathbb{T}$-model M, we have:

$$[\![\vec{x}.\varphi]\!]_M \cong \mathbf{PMod}\,\mathbb{T}(\langle \vec{x}.\varphi \rangle_{\mathbb{T}}, M).$$

Definition

An object $A \in \mathscr{A}$ is κ-presentable if its Hom-functor

$$\mathscr{A}(A, -) \colon \mathscr{A} \to \mathbf{Set}$$

preserves κ-filtered colimits.

Theorem ([Kaw23a; Kaw24])

$\mathbb{T}$: a κ-ary partial Horn theory
TFAE for a $\mathbb{T}$-model $M \in \mathbf{PMod}\,\mathbb{T}$:

1. M is κ-presentable.
2. There exists a κ-ary Horn formula $\vec{x}.\varphi$ s.t. $M \cong \langle \vec{x}.\varphi \rangle_{\mathbb{T}}$.

Example: UDO semirings

Example ([Gol03])

A uniquely difference-ordered semiring consists of:

- a base poset $(R, \leq)$;
- total operators $+, \cdot \colon R \times R \to R$;
- constants $0, 1 \in R$;
- a partial operator $\ominus \colon R \times R \rightharpoonup R$ such that

$$b \ominus a \text{ is defined iff } a \leq b$$

which satisfy the following:

- $(R, +, \cdot, 0, 1)$ is a semiring;
- $a \leq a + b$;
- $(a + b) \ominus a = b$;
- $a + (b \ominus a) = b$ whenever $a \leq b$.

UDO semirings are algebras over posets.

Example: partial abelian groups

Example ([BH12])

A partial abelian group consists of:

- a base set A with a reflexive symmetric relation $\odot \subseteq A \times A$; (a set with commeasurability)
- a constant $0 \in A$;
- a total operator $-: A \to A$;
- a partial operator $+: A \times A \rightharpoonup A$ such that

$$a + b \text{ is defined iff } a \odot b$$

which satisfy the following:

- $a \odot 0$;
- $a \odot (-b)$ whenever $a \odot b$;
- $a \odot (b + c)$ whenever $a \odot b$, $b \odot c$, $c \odot a$;
- $(a + b) + c = a + (b + c)$ whenever $a \odot b$, $b \odot c$, $c \odot a$;
- $a + b = b + a$ whenever $a \odot b$;
- $a + 0 = a$ and $a \odot (-a) = 0$.

Definition

A monoid-graded set is a map $\mathrm{d}\colon X \to M$ from a set X to a monoid $(M, \cdot, e)$.

Example

A monoid-graded ring consists of:

- a base monoid-graded set $(X, \mathrm{d}, M, \cdot, e)$;
- a constant $1 \in X$;
- total operators $\otimes\colon X \times X \to X, \quad 0\colon M \to X, \quad -\colon X \to X$;
- a partial operator $+\colon X \times X \rightharpoonup X$ s.t. $x + y$ is defined iff $\mathrm{d}(x) = \mathrm{d}(y)$

which satisfy the following:

- $\mathrm{d}(1) = e, \quad \mathrm{d}(x \otimes y) = \mathrm{d}(x)\mathrm{d}(y), \quad \mathrm{d}(0(a)) = a, \quad \mathrm{d}(-x) = \mathrm{d}(x)$;
- $\mathrm{d}(x + y) = \mathrm{d}(x)$ whenever $\mathrm{d}(x) = \mathrm{d}(y)$;
- $(x \otimes y) \otimes z = x \otimes (y \otimes z), \quad 1 \otimes x = x = x \otimes 1$;
- $x + 0(\mathrm{d}(x)) = x, \quad x + (-x) = 0(\mathrm{d}(x))$;
- $(x + y) + z = x + (y + z)$ whenever $\mathrm{d}(x) = \mathrm{d}(y) = \mathrm{d}(z)$;
- $x + y = y + x$ whenever $\mathrm{d}(x) = \mathrm{d}(y)$;
- $(x + y) \otimes z = x \otimes z + y \otimes z$ and $z \otimes (x + y) = z \otimes x + z \otimes y$ whenever $\mathrm{d}(x) = \mathrm{d}(y)$.

Closed monomorphisms

Definition ([Kaw23a; Kaw24])

Let $\mathbb{T}$ be a κ-ary partial Horn theory over an S-sorted κ-ary signature Σ.

1. A monomorphism $A \hookrightarrow B$ in $\mathbf{PMod}\,\mathbb{T}$ is called $\mathbb{T}$-closed (or Σ-closed) if the following diagrams form pullback squares for any $f, R \in \Sigma$.

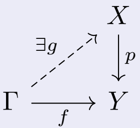
$$\begin{array}{ccc} \mathrm{Dom}(\llbracket f \rrbracket_A) & \hookrightarrow & \prod_{i<\alpha} A_{s_i} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Dom}(\llbracket f \rrbracket_B) & \hookrightarrow & \prod_{i<\alpha} B_{s_i} \end{array} \qquad \begin{array}{ccc} \llbracket R \rrbracket_A & \hookrightarrow & \prod_{i<\alpha} A_{s_i} \\ \downarrow & \lrcorner & \downarrow \\ \llbracket R \rrbracket_B & \hookrightarrow & \prod_{i<\alpha} B_{s_i} \end{array}$$

2. A morphism $h\colon A \to B$ in $\mathbf{PMod}\,\mathbb{T}$ is called $\mathbb{T}$-dense (or Σ-dense) if h factors through no $\mathbb{T}$-closed proper subobject of B.

Local retracts

Definition ([Kaw23b; Kaw24])

A morphism $p\colon X \to Y$ in a category $\mathscr{A}$ is called a κ-local retraction if for every κ-presentable object $\Gamma \in \mathscr{A}$ and every morphism $f\colon \Gamma \to Y$, there exists a morphism $g\colon \Gamma \to X$ such that $p \circ g = f$.



A κ-local retraction is also called a κ-pure quotient in [AR04].

Definition ([Kaw23b; Kaw24])

Let $U\colon \mathscr{A} \to \mathscr{C}$ be a functor. A morphism p in $\mathscr{A}$ is called a (U, κ)-local retraction if Up is a κ-local retraction in $\mathscr{C}$.

The ascending chain condition for categories

Example ([Kaw23b])

1. $\mathbf{Set}$, $\mathbf{Pos}$, and $\mathbf{Ab}$ satisfy ACC.
2. $\mathbf{Ring}$ and $\mathbf{Lat_{0,1}}$ do not satisfy ACC.
3. $\mathbf{Set}^S$ satisfies ACC $\Leftrightarrow$ S is finite. (S: a set)
4. $S/\mathbf{Set}$ satisfies ACC $\Leftrightarrow$ S is finite. (S: a set)
5. $\mathbf{Set}^{\rightarrow}$ satisfies ACC.
6. $\mathbf{Set}^{\cdot \rightrightarrows \cdot}$ does not satisfy ACC.
7. $\mathbf{Set}^{\omega}$ satisfies ACC.
8. $\mathbf{Set}^{\omega^{\mathrm{op}}}$ does not satisfy ACC.
9. The category $\mathbf{URel}$ of sets with a unary relation satisfies ACC.
10. The category $\mathbf{BRel}$ of sets with a binary relation does not satisfy ACC.