On the iterated analogue of the fundamental theorem of homomorphisms

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⇜ slides

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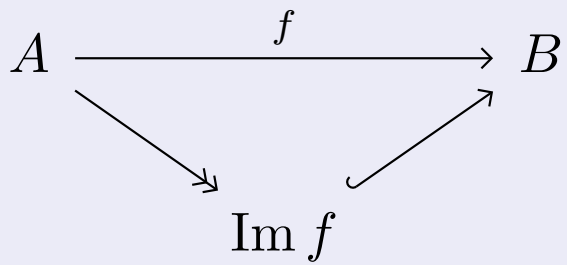
1 Regular epimorphisms

2 Generalized regular epimorphisms

3 Further topics

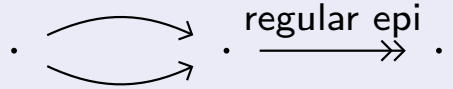
Recall

Every homomorphism $f\colon A \to B$ between groups can be decomposed into a surjective hom. followed by an injective hom.



in $\mathbf{Grp}$

Regular epimorphisms = morphisms being the coequalizer of some parallel pair of morphisms.



(eg. regular epi = surjective hom. in $\mathbf{Grp}$)

Monomorphisms = morphisms s.t. $f \circ -$ is injective.

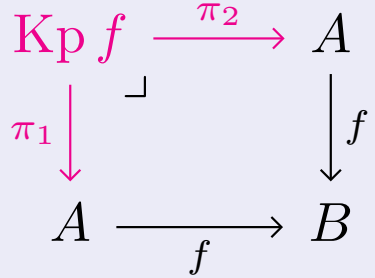


(eg. mono = injective hom. in $\mathbf{Grp}$)

The fundamental homomorphism theorem

$$A/\operatorname{Ker} f \cong \operatorname{Im} f \qquad (f\colon A \to B)$$

The kernel pair of f:

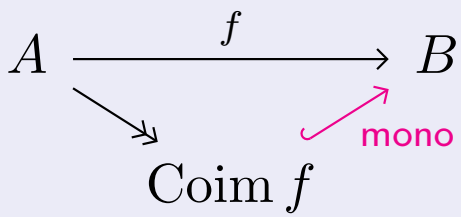


(pullback)

The coimage of f:

$$\operatorname{Kp} f \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} A \twoheadrightarrow \operatorname{Coim} f \qquad \text{(coequalizer)}$$

The fundamental homomorphism theorem (categorically)



This holds in any categories of equational algebras.

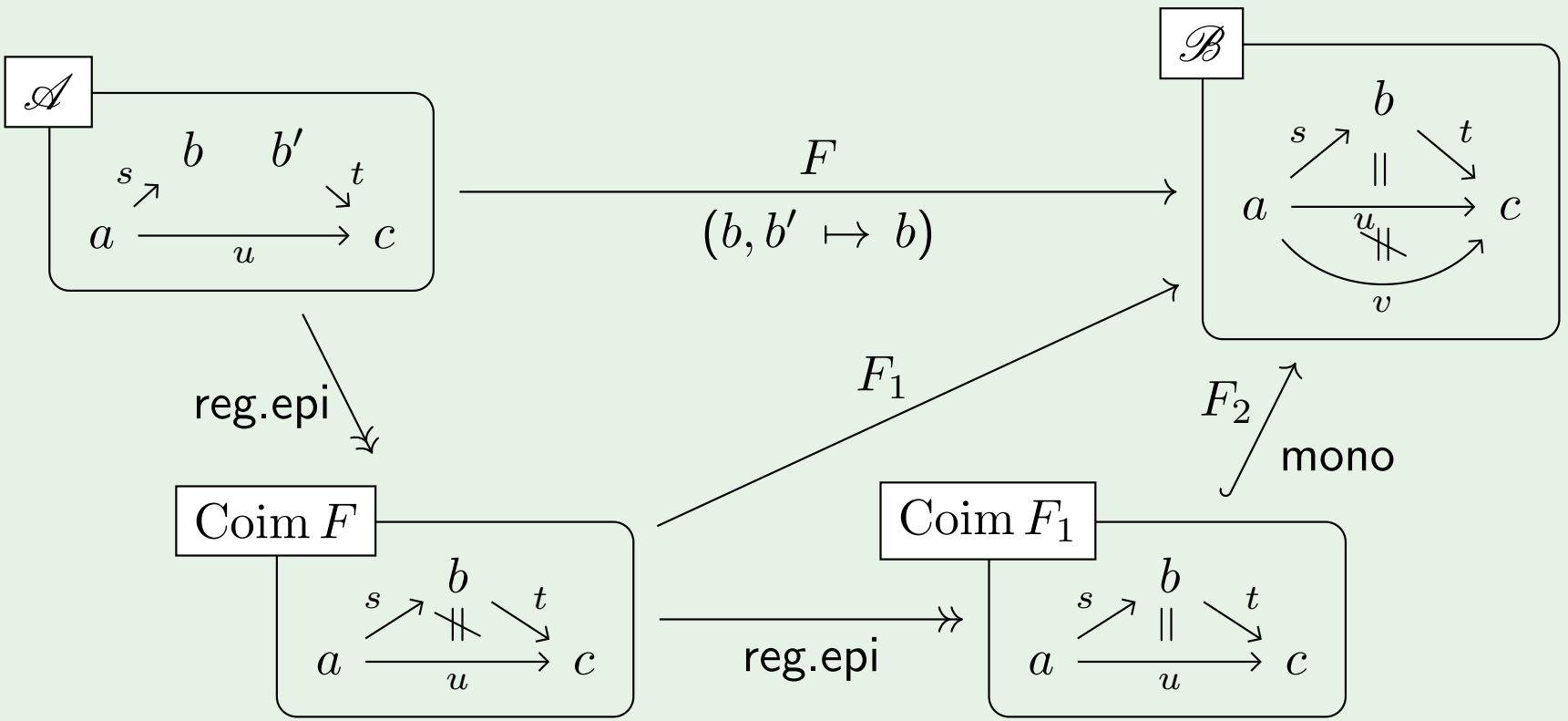
Question

Does the fundamental hom. thm. hold in any category?

$\rightsquigarrow$ No.

Counter example

In **Cat**,

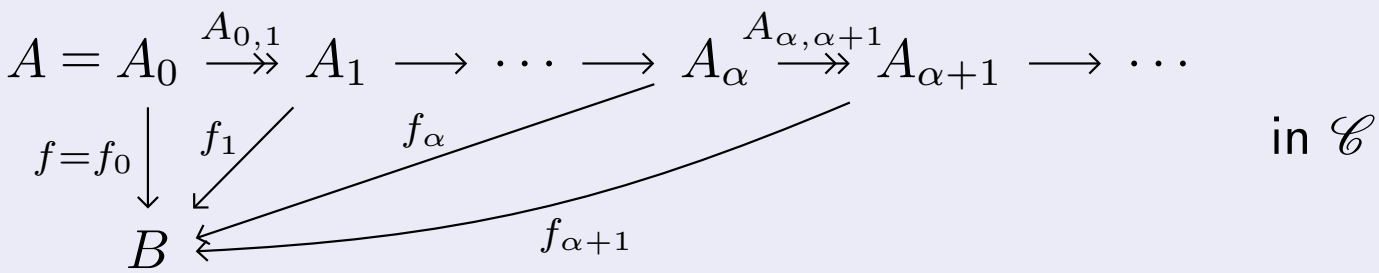


Definition

A (regular) decomposition of $A \xrightarrow{f} B$ in $\mathscr{C}$ consists of:

- A functor $\mathbf{Ord} \xrightarrow{A_\bullet} \mathscr{C}$ with $A_0 = A$;
- A cocone $f_\bullet = (A_\alpha \xrightarrow{f_\alpha} B)$ over $A_\bullet$ with $f_0 = f$

satisfying ($\forall\alpha$, $A_{\alpha,\alpha+1}$: regular epi) and ($\forall\gamma$: limit type, $A_\gamma \cong \mathrm{Colim}_{\alpha<\gamma} A_\alpha$).

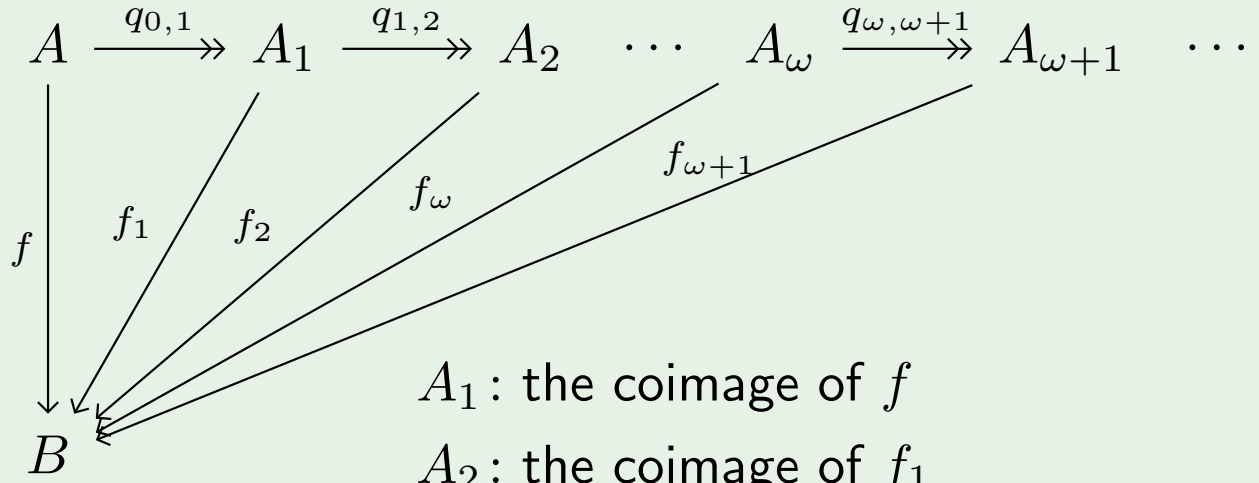


in $\mathscr{C}$

Definition

1. A decomp. $(A_\bullet, f_\bullet)$ stabilizes at $\alpha \overset{\text{def}}{\Leftrightarrow} f_\beta$: mono $(\alpha \leq \forall\beta)$ $(\Leftrightarrow f_\alpha$: mono$)$
2. The smallest α above is called the length of $(A_\bullet, f_\bullet)$. (if exists)

$\mathscr{C}$: a category with kernel pairs and colimits.



in $\mathscr{C}$

A_1 : the coimage of f

A_2 : the coimage of f_1

A_ω : the colimit of the chain $(A_n)_{n<\omega}$

$A_{\omega+1}$: the coimage of f_ω

$\rightsquigarrow$ This gives a decomposition of f. (canonical decomposition)

Definition

1. The canonical decomposition number $\sigma(f) :=$ the length of the canonical decomposition of f. If it doesn't stabilizes, $\sigma(f)$ is left to be undefined.
2. If $\sigma(f)$ is defined for every f in $\mathscr{C}$, the global canonical decomposition number $\sigma(\mathscr{C}) := \min\{\alpha \mid \sigma(f) < \alpha \text{ for every } f \text{ in } \mathscr{C}\}$

Remark

1. $\sigma(\mathscr{C}) = 0 \iff \mathscr{C} = \varnothing$.
2. $\sigma(\mathscr{C}) \leq 1 \iff \sigma(f) = 0\ (\forall f) \iff f$: monic $(\forall f)$.
3. $\sigma(\mathscr{C}) \leq 2 \iff \sigma(f) \leq 1\ (\forall f) \iff$ "The fund. hom. thm. holds in $\mathscr{C}$."
4. $\sigma(\mathscr{C}) \leq \omega \iff \sigma(f)$: finite $(\forall f)$.

Example

1. $\sigma(\mathbf{Set}) = \sigma(\mathbf{Grp}) = 2$.
2. $\sigma(\mathbf{Cat}) = 3$.
3. $\sigma(n\text{-}\mathbf{Cat}) = n + 2$.

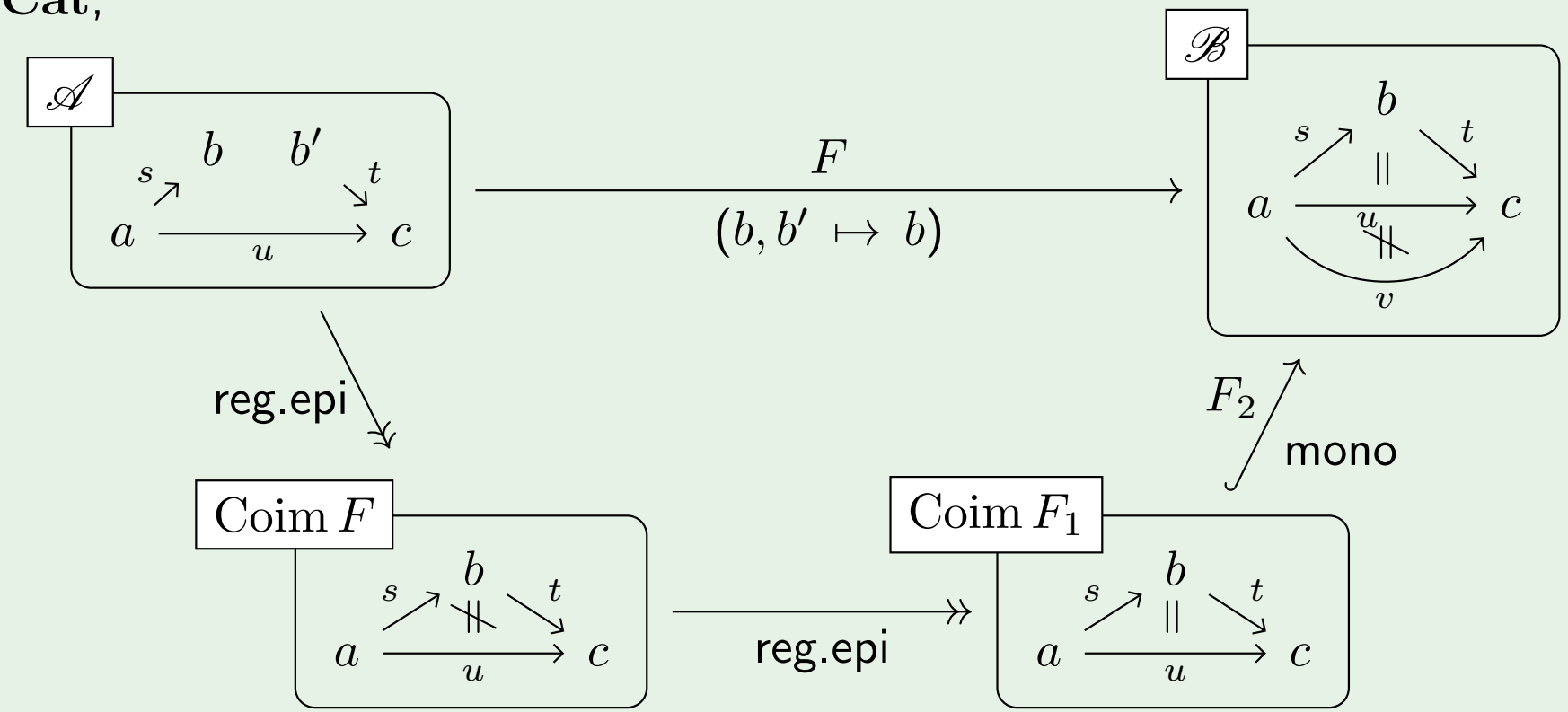
Theorem [GU71]

$\mathscr{C}$: locally κ-presentable $\implies$ $\sigma(f)$ is defined and $\sigma(f) \leq \kappa$ holds. $(\forall f$ in $\mathscr{C})$
(Equivalently, $\sigma(\mathscr{C}) \leq \kappa + 1$)

Minimum decomposition problem

Recall (a canonical decomposition in $\mathbf{Cat}$)

In $\mathbf{Cat}$,



Question

Is there a shorter decomposition of F than the canonical one in $\mathbf{Cat}$? That is, can F be decomposed as $\cdot \xrightarrow{\text{reg.epi}} \cdot \xhookrightarrow{\text{mono}} \cdot$?

Minimum decomposition problem

Definition

The minimum decomposition number $\delta(f)$:= the shortest possible length of decompositions of f. (if at least one decomposition exists)

Question (restated)

Does $\sigma(f) = \delta(f)$ hold?

Theorem (Canonical vs minimum [KN])

In a category with kernel pairs and colimits,

1. $\sigma(f)$ is defined $\iff$ $\delta(f)$ is defined.
2. Whenever they are defined, $\sigma(f) = \delta(f)$ holds.

That is, the canonical (regular) decomposition is minimum.

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Observation

In a category with kernel pairs,

q: regular epi. $\iff$ q: the coequalizer of the kernel pair of q.

- "coequalize the kernel pair of q" $\iff$ coequalize any pair of morphisms that coequalized by q. $\cdots$ $(\star)$ (*the kernel condition associated with* q)

$\therefore$ q: regular epi. $\iff$ q: the most universal among morphisms satisfying $(\star)$.

In a category with binary coproducts,

$$X \overset{u}{\underset{v}{\rightrightarrows}} A \text{ is coequalized by } \begin{array}{c} A \\ \downarrow{\scriptstyle q} \\ B \end{array} \iff \begin{array}{ccc} X+X & \xrightarrow{(u,v)} & A \\ {\scriptstyle \nabla}\downarrow & & \downarrow{\scriptstyle q} \\ X & \overset{\exists}{\dashrightarrow} & B \end{array}$$

$$A \xrightarrow{f} C \text{ satisfies } (\star) \iff \forall X, \begin{array}{ccc} X+X & \xrightarrow{\forall} & A \\ {\scriptstyle \nabla}\downarrow & & {\scriptstyle q}\downarrow \quad \searrow{\scriptstyle f} \\ X & \xrightarrow{\forall} & B \\ & \overset{\exists}{\searrow} & \quad C \end{array}$$

Idea

Replace $\{X + X \xrightarrow{\nabla} X\}_{X\,:\,\text{obj}}$ with another class of epimorphisms.

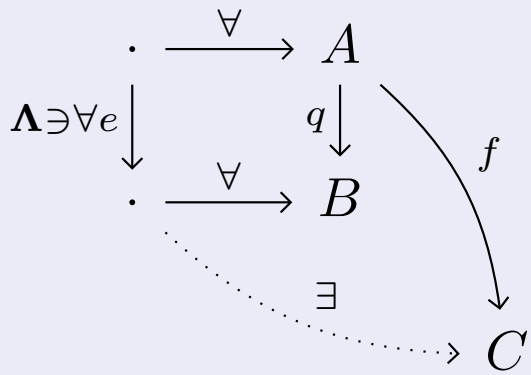
$\mathbf{\Lambda}$-regular epimorphisms

$\mathbf{\Lambda}$: a class of epimorphisms. Fix $A \xrightarrow{q} B$.

Definition [KN]

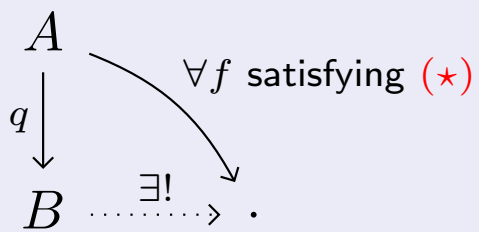
$A \xrightarrow{f} C$ satisfies $(\star)$
(the $\mathbf{\Lambda}$-*kernel condition assoc. with* q)

$\overset{\text{def}}{\Leftrightarrow}$



Definition [KN]

q: $\mathbf{\Lambda}$-regular epi $\overset{\text{def}}{\Leftrightarrow}$



Example I

Example

$\mathbf{Pos}$: the category of posets and order-preserving maps.

$$\mathbf{\Lambda} := \left\{ \boxed{\bullet \quad \bigstar} \xrightarrow{\iota} \boxed{\bullet < \bigstar} \quad \text{in } \mathbf{Pos} \right\}$$

$\rightsquigarrow$ $\mathbf{\Lambda}$-regular epis $=$ *surjective* order-preserving maps ($=$ epis)

Example

$\mathbf{Cat}$: the category of small categories and functors.

$$\mathbf{\Lambda} := \left\{ \boxed{\bullet \to \bigstar} \xrightarrow{L} \boxed{\bullet \cong \bigstar} \quad \text{in } \mathbf{Cat} \right\}$$

$\rightsquigarrow$ $\mathbf{\Lambda}$-regular epis $=$ *strict localizations*

Example II

Example

$\mathbf{Top}$: the category of topological spaces and continuous maps.

$$\mathbf{\Lambda} := \left\{ K \xrightarrow{!_K} 1 \quad \text{in } \mathbf{Top} \right\}_{K:\text{connected space}}$$

$\rightsquigarrow q$: $\mathbf{\Lambda}$-regular epi $\iff$ $q^{-1}(x)$: connected $(\forall x)$. $(=$ *monotone quotient*$)$

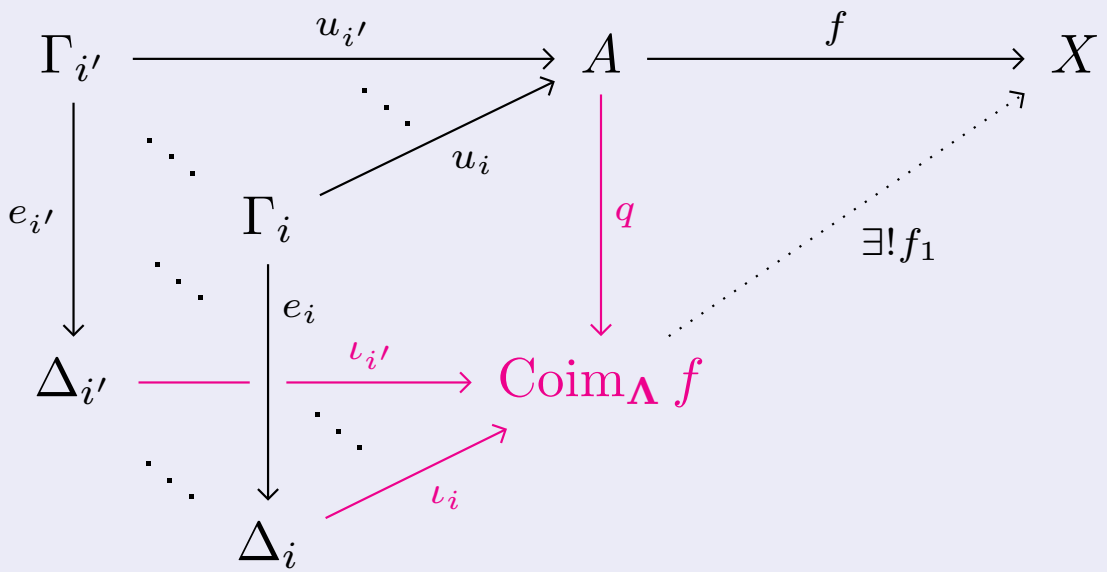
Coimages in terms of $\mathbf{\Lambda}$-regular epis

Definition

Let $A \xrightarrow{f} X$.

$$\mathcal{S} \coloneqq \left\{ (e_i, u_i) \;\middle|\; \begin{array}{ccccc} \Gamma_i & \xrightarrow{u_i} & A & \xrightarrow{f} & X \\ \mathbf{\Lambda} \ni e_i \downarrow & & & \nearrow_{\exists} & \\ \Delta_i & & & & \end{array} \right\}$$

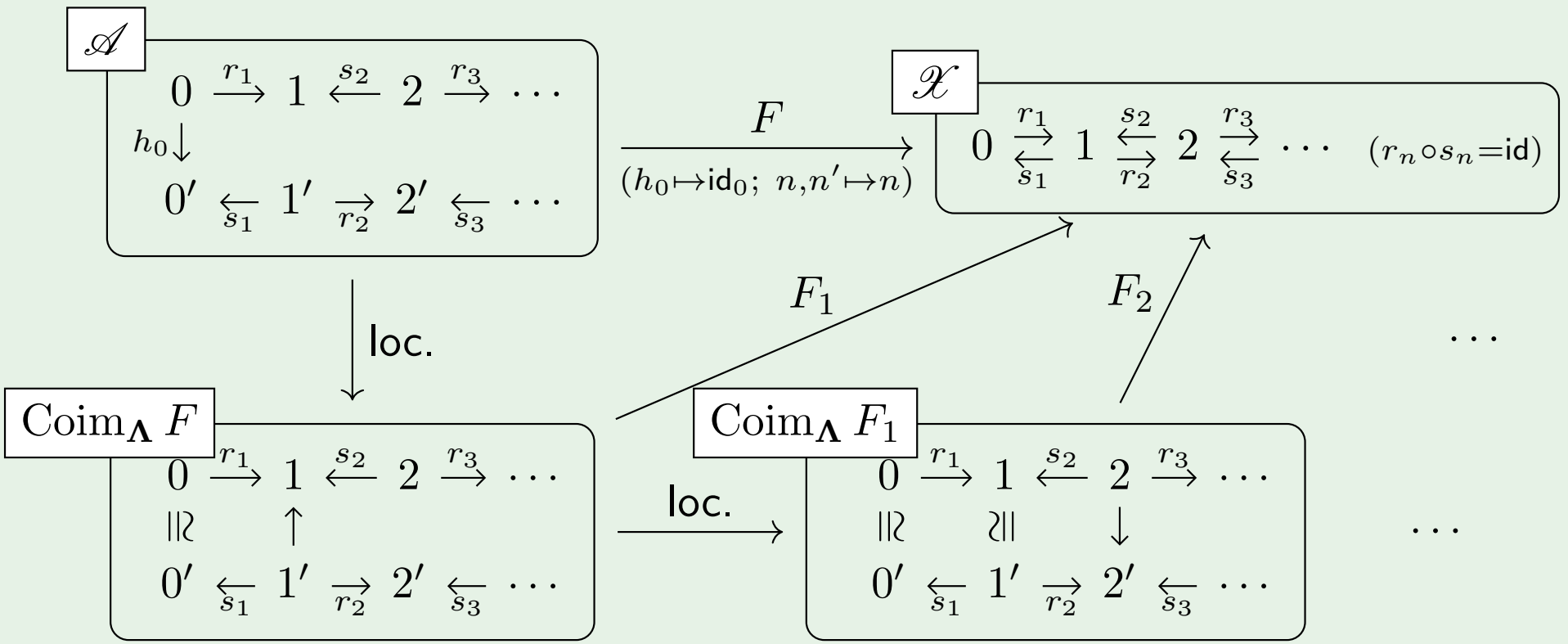
Then, $\mathrm{Coim}_{\mathbf{\Lambda}}\, f \coloneqq \mathrm{Colim}\, \mathcal{S}$. (multiple pushout)



q becomes $\mathbf{\Lambda}$-regular epi!

Example

In $\mathbf{Cat}$ with $\mathbf{E} := \{\text{strict localizations}\}$,



$\rightsquigarrow\ \sigma_{\mathbf{E}}(F) = \omega.$

Theorem (Small object argument)

$\mathscr{C}$: locally κ-presentable; $\mathbf{\Lambda}$: small; $\mathrm{dom}\, f, \mathrm{cod}\, f$: κ-presentable ($\forall f \in \mathbf{\Lambda}$)
$\Longrightarrow$ $\sigma_{\mathbf{E}}(\mathscr{C}) \leq \kappa + 1$, where $\mathbf{E} := \mathbf{Reg}(\mathbf{\Lambda})$.

Corollary

$\sigma_{\mathbf{E}}(\mathbf{Cat}) = \omega + 1$, where $\mathbf{E} :=$ {strict localizations}.

Minimum decomposition problem

Let $\mathbf{E} \coloneqq \mathbf{Reg}(\mathbf{\Lambda})$.

We can also generalize $\left(\begin{array}{ccc} \text{regular decomposition} & \rightsquigarrow & \mathbf{E}\text{-decomposition} \\ \text{minimum dec.num. } \delta(f) & \rightsquigarrow & \delta_{\mathbf{E}}(f) \end{array} \right)$.

Then, $\sigma_{\mathbf{E}}$ and $\delta_{\mathbf{E}}$ still coincide:

Theorem (Canonical vs minimum [KN])

Let $\mathscr{C}$: locally small and cocomplete, $\mathbf{E} \coloneqq \mathbf{Reg}(\mathbf{\Lambda})$ with $\mathbf{\Lambda}$: small.

1. $\sigma_{\mathbf{E}}(f)$ is defined $\iff$ $\delta_{\mathbf{E}}(f)$ is defined.
2. Whenever they are defined, $\sigma_{\mathbf{E}}(f) = \delta_{\mathbf{E}}(f)$ holds.

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Further topics

We actually do more ...

- We generalize the coimage-factorizations further to locally orthogonal factorizations (in the sense of [Tho83]).
  $\rightsquigarrow$ New example: *monadic tower* and *monadic length*.
- We give two concrete ways to determine the global regular-dec. num. $\delta(\mathscr{C})$ for a given locally presentable category $\mathscr{C}$.
  - An approach via partial Horn theories.
  - An approach via generalized algebraic theories.

  $\rightsquigarrow$ Either of them provides a complete proof for $\delta(n\text{-}\mathbf{Cat}) = n + 2$, etc.

Thank you!

slides $\rightsquigarrow$



This talk is based on the first half of [KN].

References

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Example

$\mathscr{C}$: the (l.f.p.) category of sets M with countably many partial constants $b, c_0, c_1, c_2, \ldots$ such that:

$$\left(c_n = b \longmapsto\!\!\!\!\!\!- \; c_{n+1}{\downarrow}\right)_{n\geq 0}$$

$\rightsquigarrow \sigma(\mathscr{C}) = \omega + 1$

Example

$\mathscr{C}$: the (l.f.p.) category of sets M with countably many partial constants $b, c_0, c_1, c_2, \ldots$ such that:

$$\top \longmapsto\!\!\!\!\!\!- \; b{\downarrow}$$

$$\left(c_n{\downarrow} \longmapsto\!\!\!\!\!\!- \; c_{n+1}{\downarrow}\right)_{n\geq 0}$$

$$\left(c_{n+1} = b \longmapsto\!\!\!\!\!\!- \; c_n{\downarrow}\right)_{n\geq 0}$$

$\rightsquigarrow \sigma(\mathscr{C}) = \omega$